CSC236 tutorial exercises, Week #1
sample solutions

1. Define $P(n)$ as:
$$\sum_{i=0}^{i=n} 2^i = 2^{n+1}$$
   (a) Prove that $P(115)$ implies $P(116)$.

   proof: Assume $P(115)$, that is $\sum_{i=0}^{i=115} 2^i = 2^{116}$. I must now show that $P(116)$ follows. Notice that
$$\begin{aligned} \sum_{i=0}^{i=116} 2^i &= \left[\sum_{i=0}^{i=115} 2^i\right] + 2^{116} \qquad \# \text{ regrouping} \\ &= 2^{116} + 2^{116} \qquad \# \text{ by } P(115) \\ &= 2^{117} \quad \blacksquare \end{aligned}$$

   It is also possible to note that $P(115)$ is false, and an implication with a false hypothesis is always true (vacuous truth).

   (b) Is $P(n)$ true for every natural number n? Explain why, or why not.

   solution: $P(n)$ is false for **every** natural number n. Because of this it is impossible to verify a base case, so the correct induction step (see above) does not establish a proof.

2. Use induction to prove that $\forall n \in \mathbb{N}$, $8^n - 1$ is a multiple of 7.

   proof by simple induction: Define $P(n) : \exists k \in \mathbb{N}, 8^n - 1 = 7k$. I will prove $\forall n \in \mathbb{N}, P(n)$.

   base case: $8^0 - 1 = 0 = 7 \times 0$, which verifies $P(0)$.

   inductive step: Let $n \in \mathbb{N}$ and assume $P(n)$, let k be such that $8^n - 1 = 7k$. Let $k' = 8k + 1$. I will show that $8^{n+1} - 1 = 7k'$.
$$\begin{aligned} 8^{n+1} - 1 &= 8(8^n - 1) + 7 \\ &= 8(7k) + 7 \qquad \# \text{ by } P(n) \\ &= 7(8k+1) = 7k' \quad \blacksquare \qquad \# \text{ by choice of } k' \end{aligned}$$

   So $P(n) \implies P(n+1)$ for arbitrary n.

3. Use induction to prove that for every power of 7, there is a power of 3 with the same units digit.

   proof by simple induction: Define $P(n) : \exists m \in \mathbb{N}, 7^n \equiv 3^m \bmod 10$. I must prove $\forall n \in \mathbb{N}, P(n)$.

base case: $7^0 = 1 = 3^0$, which verifies $P(0)$.

inductive step: Let $n \in \mathbb{N}$. Assume $P(n)$, and let m b e such that

$$7^n \equiv 3^m \bmod 10$$

Let $m' = m + 3$. I will show $P(n+1)$ follows, that is

$$7^{n+1} \equiv 3^{m+3} \bmod 10$$

Note that $3^3 \equiv 7 \bmod 10$, so $3^3 k \equiv 7k \bmod 10$ for any k. See Example 2.18 in the CSC165 course notes.

$$\begin{aligned} 7 \times 7^n &\equiv 3^3 \times 7^n \bmod 10 && \text{\# by Example 2.18} \\ &\equiv 3^3 \times 3^m \bmod 10 && \text{\# by I.H. and Example 2.18 again} \\ &\equiv 3^{m+3} \bmod 10 \quad \blacksquare \end{aligned}$$

note: This could also be proven by combining...

- the proof from the lecture notes that all powers of 7 have a units digit in $\{1, 3, 7, 9\}$
- a proof that for each $u \in \{1, 3, 7, 9\}$ there exists an m such that 3^m has units digit u. This is as simple as providing an example for each digit, i.e. $3^0 = 1, 3^1 = 3, 3^2 = 9, 3^3 = 27$.

4. Consider an alternative to our familiar inductive proof structure in which we prove the following:

$$P(0) \tag{1}$$
$$P(1) \tag{2}$$
$$\forall n, m \in \mathbb{N}, P(n) \wedge P(m) \implies P(n+m) \tag{3}$$

Is this a valid proof that P holds for all natural numbers?

(a) Use simple induction with the facts above to prove $\forall n \in N, P(n)$.

proof: We will prove this using simple induction on n.

inductive step: Let $n \in \mathbb{N}$ and assume $P(n)$.
$P(n+1)$ follows from using (3) to combine $P(n)$ (the I.H.) with $P(1)$ (2).

basis: The base case of $P(0)$ is given by (1). $\blacksquare$

(b) If we omit claim (3) above, obviously we can't conclude anything more profound than $P(0) \wedge P(1)$. But what numbers can we conclude that P holds for if we...

i. Omit (1)?

solution: $\mathbb{N}^+$ (i.e. all naturals except 0). Our inductive step above doesn't use (1) so we still have $P(1) \implies P(2)$, $P(2) \implies P(3)$, etc.

ii. Omit (2)?

solution: We can only conclude $P(0)$. With just $P(0)$, we can't use (3) to generate any further cases.

iii. Replace (1) and (2) with $P(2)$ and $P(3)$?

solution: All numbers of the form $2j + 3k$ for $j, k \in \mathbb{N}$. (This happens to be all the natural numbers > 1. How would we prove this fact? Stay tuned for next week!)