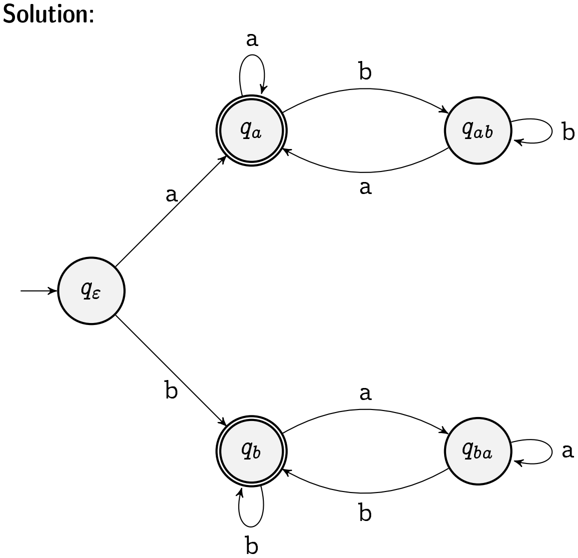
CSC236 tutorial exercises, Week #10
sample solutions

1. Let BOOKENDS $= \{x \in \{a, b\}^* \mid |x| > 0 \wedge x[0] = x[-1]\}$, i.e. the language of non-empty strings where the first symbol matches the last symbol. Design (i.e. draw) a DFA that accepts BOOKENDS.

   Solution:

   Note that our machine must accept the strings a and b.

   There's nothing special about the names we used to label the nodes of the DFA, they were just chosen to help remind us of the 'meaning' of each state.

2. Consider the following formal definition of a DFA, $M = (Q, \Sigma, \delta, s, F)$, where

   - $Q = \{q_0, q_1, q_2\}$
   - $\Sigma = \{a, b\}$
   - $s = q_0$
   - $F = \{q_2\}$

   And the transition function δ is given by the following table (with states on the left, and symbols on top):

δ	a	b
q_0	q_1	q_0
q_1	q_2	q_1
q_2	q_2	q_2

(a) Give a short English description of $\mathcal{L}(M)$, the language accepted by M.

Solution:
$\mathcal{L}(M)$ is the language of strings having at least two occurrences of the symbol a.

(b) Complete the following predicate definition by defining invariants for each of M's three states. Your invariants should imply the correctness of your description from part (a). (Recall that δ^* is the extended transition function):

$$P(x) : \delta^*(s,x) = \begin{cases} q_0 & \text{if } \_\_\_\_\_\_ \\ q_1 & \text{if } \_\_\_\_\_\_ \\ q_2 & \text{if } \_\_\_\_\_\_ \end{cases}$$

The conditions on x that you fill in should be exhaustive (so that the "if"s are equivalent to "iff").

Solution:

$$P(x) : \delta^*(s,x) = \begin{cases} q_0 & \text{if x has no a's} \\ q_1 & \text{if x has exactly one a} \\ q_2 & \text{if x has two or more a's} \end{cases}$$

(c) Use induction to prove $\forall x \in \{a,b\}^*, P(x)$.

Solution:
Basis: ε has no a's, and $\delta^*(s,\varepsilon) = s = q_0$, so $P(\varepsilon)$ holds.
Inductive step: Assume $P(x)$ for an arbitrary string x. I want to show $P(xa)$ and $P(xb)$.
By the δ-table above, I can see that $\delta^*(s,xb) = \delta^*(s,x)$ (since for every state q, the b transition just loops back to q). Furthermore, xb has the same properties as x with respect to number of a's, so $P(xb)$ holds.
By the inductive hypothesis

$$\delta^*(s,x) = \begin{cases} q_0 & \text{if x has no a's} \\ q_1 & \text{if x has exactly one a} \\ q_2 & \text{if x has two or more a's} \end{cases}$$

Therefore

$$\delta^*(s,xa) = \delta(\delta^*(s,x),a) = \begin{cases} q_1 & \text{if x has no a's} \\ q_2 & \text{if x has exactly one a} \\ q_2 & \text{if x has two or more a's} \end{cases}$$

In other words

$$\delta^*(s,xa) = \begin{cases} q_1 & \text{if xa has one a} \\ q_2 & \text{if xa has exactly two a's} \\ q_2 & \text{if xa has more than two a's} \end{cases}$$

As required. (Note that, by construction, it's impossible that xa has no a's.)

3. Suppose we have a DFA, $M = (Q, \Sigma = \{0, 1\}, \delta, s, F)$, such that for every state $q \in Q$, $\delta(q, 0) = \delta(q, 1)$. What can we conclude about $\mathcal{L}(M)$, the language accepted by M?

   Solution:
   Informally, this tells us that the state reached by the machine for any string x is determined entirely by $|x|$ (i.e. the length of the string).

   Thus we can say that for every $n \in \mathbb{N}$, either $\Sigma^n \subseteq \mathcal{L}(M)$, or $\Sigma^n \cap \mathcal{L}(M) = \emptyset$. i.e. either all strings of a given length are in the language, or none of them are.