CSC236 Winter 2020
Assignment #1: induction
Solutions

1. Define function f recursively as follows:

$$f(n) = \begin{cases} 1 & \text{if } n \leq 1 \\ n \cdot f(n-2) & \text{if } n > 1 \end{cases}$$

Use induction to prove that for all even $n \in \mathbb{N}$, $f(n) = 2^{n/2}(n/2)!$.

Solution:
Define $P(i) : f(2i) = 2^i i!$. I will use simple induction to prove $\forall i \in \mathbb{N}, P(i)$, which is equivalent to the claim that $f(n) = 2^{n/2}(n/2)!$ for all even n.[1]
Inductive Step: Let $i \in \mathbb{N}$ and assume $P(i)$.
I will show that $P(i+1)$ follows, i.e. that $f(2(i+1)) = 2^{i+1}(i+1)!$.

$$\begin{aligned} f(2(i+1)) &= f(2i+2) \\ &= (2i+2) \cdot f(2i) && \text{\# By definition of } f(n)\text{, since } 2i+2 > 1 \\ &= (2i+2) \cdot 2^i i! && \text{\# By I.H.} \\ &= 2(i+1) \cdot 2^i i! \\ &= (i+1) \cdot 2^{i+1} i! \\ &= 2^{i+1}(i+1)! && \text{\# By definition of factorial} \end{aligned}$$

Thus, $P(i+1)$.
Base Case: $f(2 \cdot 0) = f(0) = 1 = 2^0 0!$, thus $P(0)$.
Therefore, by the principle of induction, $\forall i \in \mathbb{N}, P(i)$, as required. ■

Comment: A change of variable simplified this proof. I could also have done induction on the argument to f itself, but then I would have needed to establish in the I.S. that $P(n) \implies P(n+2)$.

[1] I chose i (rather than n) as the name of the variable for my predicate and in the induction step just to avoid confusion with the use of n in the statement of the question. However, I could just as easily have used n - it would not have changed the validity of my proof.

2. What happens when the fall of the nth domino implies the fall of the *previous* one? Suppose we have proven the following facts with respect to some predicate $P(n)$:[2]

$$P(1) \tag{1}$$
$$\forall n \in \mathbb{N}^+, P(n) \implies P(n-1) \tag{2}$$
$$\forall n \in \mathbb{N}, P(n) \implies P(2n) \tag{3}$$

In this question, you will show that, taken together, these three statements comprise a valid proof that P holds for all natural numbers.

(a) Use complete induction to prove that $\forall n \in \mathbb{N}, P(n)$.

Solution Let $n \in \mathbb{N}$, and assume $P(k)$ holds for all $k < n$.

Case 1: $n = 0$. $P(n)$ follows from applying (2) to (1).

Case 2: $n = 1$. $P(n)$ is given by (1).

Case 3: $n > 1$ **and** n **is even.** Then $n/2$ is a natural number less than n, so $P(n/2)$ by the I.H.
$P(n)$ follows from applying (2) to $P(n/2)$.

Case 4: $n > 1$ **and** n **is odd.**

$$1 < n$$
$$n + 1 < 2n$$
$$\frac{n+1}{2} < n$$

Furthermore, $\frac{n+1}{2}$ is a natural number, since n is odd. Therefore, $P(\frac{n+1}{2})$ follows from the I.H. Applying (3), we can derive $P(n+1)$. Finally, applying (2) to this result, we get $P(n)$.

$P(n)$ holds in all cases. ■

Comment: Another approach would be to use (1) and (3) to show that P holds for all powers of 2. From there, we can claim that we can show $P(n)$ for any n because P holds for the next-highest power of 2 above n, and we can eventually descend to n from there by repeatedly applying rule (2). This is a reasonable argument, but the "we can eventually get down to n" portion would need to be formalized as a separate induction proof.

(b) If we failed to prove (3), but kept the other two statements, what values would we be able to conclude that P holds for? Repeat for (2) and (1).

[2]Where $\mathbb{N}^+$ denotes the positive natural numbers, i.e. $\mathbb{N} - \{0\}$.

Solution Without (3) we can only conclude $P(0) \wedge P(1)$.
Without (2), we can conclude that $P(n)$ holds for all powers of 2.
Without (1), we cannot conclude that $P(n)$ holds for any n.

3. Let $\mathcal{S}$ be the smallest set of strings defined by:

   (a) $\mathrm{u} \in \mathcal{S}$

   (b) if $s \in \mathcal{S}$ then $\mathrm{y}s \in \mathcal{S}$

   (c) if $s \in \mathcal{S}$ then $s\mathrm{h} \in \mathcal{S}$

   (d) if $s_1, s_2 \in \mathcal{S}$ then $s_1 s_2 \in \mathcal{S}$

   Use structural induction to prove that no strings in $\mathcal{S}$ contain the substring yh. **Hint:** It may help to strengthen your induction hypothesis.

   Solution (Note: For convenience, this proof uses the Python-like notation $s[i]$ to denote the ith character of string s, with $s[-1]$ denoting the last character.)
   Define $P(s)$: s does not contain the substring yh, $s[0] \neq \mathrm{h}$, and $s[-1] \neq \mathrm{y}$, and s is non-empty.[3]
   I will use structural induction to prove $\forall s \in \mathcal{S}, P(s)$.
   Base Case: $P(\mathrm{u})$ is clearly true by inspection.
   Inductive Step:
   Part 1 (rule b): Let $s_1 \in \mathcal{S}$ and assume $P(s_1)$. Consider $s = \mathrm{y}s_1$. Then we have

   - $s[0] = \mathrm{y} \neq \mathrm{h}$, by construction
   - $s[-1] \neq \mathrm{y}$ by our inductive hypothesis, since s has s_1 as a (non-empty) suffix
   - s does not contain yh. Since s_1 does not contain yh, s cannot contain the substring unless it begins at index 0. However, this is impossible, since $s[1] = s_1[0] \neq \mathrm{h}$.
   - s is non-empty

   Thus, $P(s)$.

   Part 2 (rule c): Let $s \in \mathcal{S}$ and assume $P(s_1)$. Consider $s = s_1\mathrm{h}$. Then we have

   - $s[0] = s_1[0] \neq \mathrm{h}$, by I.H.
   - $s[-1] \neq \mathrm{y}$, since the last character is h, by construction.
   - s is non-empty

[3] This last condition is not essential, but it avoids the awkwardness of talking about $s[0]$ for a string s which is potentially empty.

- s does not contain yh. Since s_1 does not contain the substring, the only way s can contain it is if it occurs as the last two characters. However, $s[-2] = s_1[-1] \neq y$ (by I.H.), so this is not possible.

Thus, $P(s)$.

Part 3 (rule d): Let $s_1, s_2 \in \mathcal{S}$ and assume $P(s_1) \wedge P(s_2)$. Consider $s = s_1 s_2$ Then we have

- $s[0] = s_1[0] \neq \text{h}$, by I.H.
- $s[-1] = s_2[-1] \neq \text{y}$, by I.H.
- s is non-empty, since s_1 and s_2 are both non-empty by I.H.
- s does not contain yh. Since the substring occurs neither in s_1 nor s_2, the only way it can occur in s is at the point where s_1 and s_2 meet, but this would require $s_1[-1] = \text{y}$ and $s_2[0] = \text{h}$, neither of which is allowed by the I.H.

Thus, $P(s)$.

Our predicate applies to the simplest element of $\mathcal{S}$, and is preserved by all the rules for making more complicated elements, so $\forall s \in \mathcal{S}, P(s)$.

4. Define $A(n)$ as the smallest natural number containing exactly n substrings in its decimal representation which are prime numbers.
For example, $A(2) = 13$, because the string '13' contains the prime numbers 3 and 13 itself (and is smaller than any other number with this property, such as 31). $A(6) = 373$, corresponding to the prime numbers 3 (which appears twice), 7, 37, 73, and 373.
Prove that $A(n)$ is defined for each $n \in \mathbb{N}$. i.e. for each $n \in \mathbb{N}$, there exists a smallest natural number containing exactly n prime substrings.

Solution Let $S(n)$ denote the set of natural numbers containing exactly n prime substrings. Observe that, for a given n, if $S(n)$ is non-empty, then $A(n)$ is equal to its minimum element.

Let $n \in \mathbb{N}$ be an arbitrary natural number. I will show that $A(n)$ exists.

Case 1: $n = 0$ Then $A(n) = 0$. 0 contains no prime substrings, and is the smallest natural number.

Case 2: $n > 0$ I can construct the decimal representation of a number q with exactly n prime substrings by concatenating n '2's. Or, more formally, $q = \sum_{i=0}^{n-1} 2 \cdot 10^i$.
q has at least n prime substrings, because 2 is a prime number, and it occurs exactly n times in q's decimal representation.
Furthermore, q has no more than n prime substrings, because any other

non-empty substring of q's decimal representation will correspond to a multi-digit number such as 22, 222, 2222, etc. These are all numbers ending in 2, and greater than 2. We know they cannot be prime, because 2 is the only even prime number.

So while q is not necessarily equal to $A(n)$, it is an element of $S(n)$. Therefore $S(n)$ is non-empty. By the principle of well-ordering, this means that $S(n)$ has a smallest element, namely $A(n)$. So $A(n)$ exists in this case.

Starting from an arbitrary n, I was able to show that $A(n)$ exists. Therefore, $A(n)$ exists for all natural numbers. ■

Comment: While not necessary, it is possible to prove this result via simple induction. Informally, the argument would go as follows. If we assume that $A(n)$ is defined for some n, then we can show (via similar reasoning as used above) that $q = 10A(n) + 2$ (i.e. the number resulting from appending a '2' to the decimal representation of $A(n)$) has exactly $n + 1$ prime substrings. Therefore, $S(n + 1)$ is non-empty, and so by PWO, $A(n + 1)$ exists.