1. Define function $f$ recursively as follows:

$$f(n) = \begin{cases} 
1 & \text{if } n \leq 1 \\
n \cdot f(n - 2) & \text{if } n > 1 
\end{cases}$$

Use induction to prove that for all even $n \in \mathbb{N}$, $f(n) = 2^{n/2}(n/2)!$.

Solution:
Define $P(i) : f(2i) = 2^i i!$. I will use simple induction to prove $\forall i \in \mathbb{N}, P(i)$, which is equivalent to the claim that $f(n) = 2^{n/2}(n/2)!$ for all even $n$.

Inductive Step: Let $i \in \mathbb{N}$ and assume $P(i)$.
I will show that $P(i + 1)$ follows, i.e. that $f(2(i + 1)) = 2^{i+1}(i + 1)!$.

$$f(2(i + 1)) = f(2i + 2)$$
$$= (2i + 2) \cdot f(2i) \quad \# \text{ By definition of } f(n), \text{ since } 2i + 2 > 1$$
$$= (2i + 2) \cdot 2^i i! \quad \# \text{ By I.H.}$$
$$= 2(i + 1) \cdot 2^i i!$$
$$= (i + 1) \cdot 2^{i+1} i!$$
$$= 2^{i+1}(i + 1)! \quad \# \text{ By definition of factorial}$$

Thus, $P(i + 1)$.

Base Case: $f(2 \cdot 0) = f(0) = 1 = 2^0 0!$, thus $P(0)$.

Therefore, by the principle of induction, $\forall i \in \mathbb{N}, P(i)$, as required.

Comment: A change of variable simplified this proof. I could also have done induction on the argument to $f$ itself, but then I would have needed to establish in the I.S. that $P(n) \implies P(n + 2)$.

\(^1\)I chose $i$ (rather than $n$) as the name of the variable for my predicate and in the induction step just to avoid confusion with the use of $n$ in the statement of the question. However, I could just as easily have used $n$ - it would not have changed the validity of my proof.
2. What happens when the fall of the $n$th domino implies the fall of the previous one? Suppose we have proven the following facts with respect to some predicate $P(n)\,$:

\[
\begin{align*}
P(1) & \quad (1) \\
\forall n \in \mathbb{N}^+, P(n) & \implies P(n-1) \quad (2) \\
\forall n \in \mathbb{N}, P(n) & \implies P(2n) \quad (3)
\end{align*}
\]

In this question, you will show that, taken together, these three statements comprise a valid proof that $P$ holds for all natural numbers.

(a) Use complete induction to prove that $\forall n \in \mathbb{N}, P(n)$.

**Solution** Let $n \in \mathbb{N}$, and assume $P(k)$ holds for all $k < n$.

Case 1: $n = 0$. $P(n)$ follows from applying (2) to (1).

Case 2: $n = 1$. $P(n)$ is given by (1).

Case 3: $n > 1$ and $n$ is even. Then $n/2$ is a natural number less than $n$, so $P(n/2)$ by the I.H.

Case 4: $n > 1$ and $n$ is odd.

\[
\begin{align*}
1 & < n \\
n + 1 & < 2n \\
\frac{n + 1}{2} & < n
\end{align*}
\]

Furthermore, $\frac{n + 1}{2}$ is a natural number, since $n$ is odd. Therefore, $P(\frac{n + 1}{2})$ follows from the I.H. Applying (3), we can derive $P(n + 1)$. Finally, applying (2) to this result, we get $P(n)$.

$P(n)$ holds in all cases. ■

**Comment:** Another approach would be to use (1) and (3) to show that $P$ holds for all powers of 2. From there, we can claim that we can show $P(n)$ for any $n$ because $P$ holds for the next-highest power of 2 above $n$, and we can eventually descend to $n$ from there by repeatedly applying rule (2). This is a reasonable argument, but the “we can eventually get down to $n$” portion would need to be formalized as a separate induction proof.

(b) If we failed to prove (3), but kept the other two statements, what values would we be able to conclude that $P$ holds for? Repeat for (2) and (1).

\[\text{Where } \mathbb{N}^+ \text{ denotes the positive natural numbers, i.e. } \mathbb{N} - \{0\}.\]
Without (3) we can only conclude $P(0) \land P(1)$.
Without (2), we can conclude that $P(n)$ holds for all powers of 2.
Without (1), we cannot conclude that $P(n)$ holds for any $n$.

3. Let $S$ be the smallest set of strings defined by:

(a) $u \in S$
(b) if $s \in S$ then $ys \in S$
(c) if $s \in S$ then $sh \in S$
(d) if $s_1, s_2 \in S$ then $s_1s_2 \in S$

Use structural induction to prove that no strings in $S$ contain the substring $yh$. Hint: It may help to strengthen your induction hypothesis.

Solution (Note: For convenience, this proof uses the Python-like notation $s[i]$ to denote the $i$th character of string $s$, with $s[-1]$ denoting the last character.) Define $P(s) : s$ does not contain the substring $yh$, $s[0] \neq h$, and $s[-1] \neq y$, and $s$ is non-empty.

I will use structural induction to prove $\forall s \in S, P(s)$.

Base Case: $P(u)$ is clearly true by inspection.

Inductive Step:

Part 1 (rule b): Let $s_1 \in S$ and assume $P(s_1)$. Consider $s = ys_1$. Then we have

- $s[0] = y \neq h$, by construction
- $s[-1] \neq y$ by our inductive hypothesis, since $s$ has $s_1$ as a (non-empty) suffix
- $s$ does not contain $yh$. Since $s_1$ does not contain $yh$, $s$ cannot contain the substring unless it begins at index 0. However, this is impossible, since $s[1] = s_1[0] \neq h$.
- $s$ is non-empty

Thus, $P(s)$.

Part 2 (rule c): Let $s \in S$ and assume $P(s_1)$. Consider $s = s_1h$. Then we have

- $s[0] = s_1[0] \neq h$, by I.H.
- $s[-1] \neq y$, since the last character is $h$, by construction.
- $s$ is non-empty

This last condition is not essential, but it avoids the awkwardness of talking about $s[0]$ for a string $s$ which is potentially empty.
• $s$ does not contain $yh$. Since $s_1$ does not contain the substring, the only way $s$ can contain it is if it occurs as the last two characters. However, $s[-2] = s_1[-1] \neq y$ (by I.H.), so this is not possible.

Thus, $P(s)$.

Part 3 (rule d): Let $s_1, s_2 \in S$ and assume $P(s_1) \land P(s_2)$. Consider $s = s_1 s_2$. Then we have

• $s[0] = s_1[0] \neq h$, by I.H.
• $s[-1] = s_2[-1] \neq y$, by I.H.
• $s$ is non-empty, since $s_1$ and $s_2$ are both non-empty by I.H.
• $s$ does not contain $yh$. Since the substring occurs neither in $s_1$ nor $s_2$, the only way it can occur in $s$ is at the point where $s_1$ and $s_2$ meet, but this would require $s_1[-1] = y$ and $s_2[0] = h$, neither of which is allowed by the I.H.

Thus, $P(s)$.

Our predicate applies to the simplest element of $S$, and is preserved by all the rules for making more complicated elements, so $\forall s \in S, P(s)$.

4. Define $A(n)$ as the smallest natural number containing exactly $n$ substrings in its decimal representation which are prime numbers.

For example, $A(2) = 13$, because the string '13' contains the prime numbers 3 and 13 itself (and is smaller than any other number with this property, such as 31). $A(6) = 373$, corresponding to the prime numbers 3 (which appears twice), 7, 37, 73, and 373.

Prove that $A(n)$ is defined for each $n \in \mathbb{N}$, i.e. for each $n \in \mathbb{N}$, there exists a smallest natural number containing exactly $n$ prime substrings.

Solution Let $S(n)$ denote the set of natural numbers containing exactly $n$ prime substrings. Observe that, for a given $n$, if $S(n)$ is non-empty, then $A(n)$ is equal to its minimum element.

Let $n \in \mathbb{N}$ be an arbitrary natural number. I will show that $A(n)$ exists.

Case 1: $n = 0$ Then $A(n) = 0$. 0 contains no prime substrings, and is the smallest natural number.

Case 2: $n > 0$ I can construct the decimal representation of a number $q$ with exactly $n$ prime substrings by concatenating $n$ '2's. Or, more formally, $q = \sum_{i=0}^{n-1} 2 \cdot 10^i$.

$q$ has at least $n$ prime substrings, because 2 is a prime number, and it occurs exactly $n$ times in $q$'s decimal representation.

Furthermore, $q$ has no more than $n$ prime substrings, because any other
non-empty substring of $q$'s decimal representation will correspond to a multi-digit number such as 22, 222, 2222, etc. These are all numbers ending in 2, and greater than 2. We know they cannot be prime, because 2 is the only even prime number.

So while $q$ is not necessarily equal to $A(n)$, it is an element of $S(n)$. Therefore $S(n)$ is non-empty. By the principle of well-ordering, this means that $S(n)$ has a smallest element, namely $A(n)$. So $A(n)$ exists in this case.

Starting from an arbitrary $n$, I was able to show that $A(n)$ exists. Therefore, $A(n)$ exists for all natural numbers. 

**Comment:** While not necessary, it is possible to prove this result via simple induction. Informally, the argument would go as follows. If we assume that $A(n)$ is defined for some $n$, then we can show (via similar reasoning as used above) that $q = 10A(n) + 2$ (i.e. the number resulting from appending a '2' to the decimal representation of $A(n)$) has exactly $n + 1$ prime substrings. Therefore, $S(n + 1)$ is non-empty, and so by PWO, $A(n + 1)$ exists.