CSC236 Winter 2020
Assignment #1: induction
due January 30th, 3 p.m.

The aim of this assignment is to give you some practice with various flavours of induction. Be sure to read each question carefully, and use the proof technique specified. For example, if a question asks you to prove a claim via complete induction, no marks will be awarded for a correct proof using a different technique. If a question simply says to use 'induction', you may choose whatever techniques from among {simple induction, complete induction, well-ordering, structural induction} you find appropriate.

Assignments are to be completed individually and typeset in LaTeX. The `.tex` source file and rendered pdf should both be uploaded to MarkUs. For further details, see the course website: `http://www.cs.toronto.edu/~colin/236/W20/assignments/`.

1. Define function f recursively as follows:

$$f(n) = \begin{cases} 1 & \text{if } n \leq 1 \\ n \cdot f(n-2) & \text{if } n > 1 \end{cases}$$

   Use induction to prove that for all even $n \in \mathbb{N}$, $f(n) = 2^{n/2}(n/2)!$.

2. What happens when the fall of the nth domino implies the fall of the *previous* one? Suppose we have proven the following facts with respect to some predicate $P(n)$:[1]

$$P(1) \tag{1}$$
$$\forall n \in \mathbb{N}^+, P(n) \implies P(n-1) \tag{2}$$
$$\forall n \in \mathbb{N}, P(n) \implies P(2n) \tag{3}$$

   In this question, you will show that, taken together, these three statements comprise a valid proof that P holds for all natural numbers.

   (a) Use complete induction to prove that $\forall n \in \mathbb{N}, P(n)$.

[1] Where $\mathbb{N}^+$ denotes the positive natural numbers, i.e. $\mathbb{N} - \{0\}$.

(b) If we failed to prove (3), but kept the other two statements, what values would we be able to conclude that P holds for? Repeat for (2) and (1).

3. Let $\mathcal{S}$ be the smallest set of strings defined by:

   - $\mathrm{u} \in S$
   - if $s \in \mathcal{S}$ then $\mathrm{y}s \in \mathcal{S}$
   - if $s \in \mathcal{S}$ then $s\mathrm{h} \in \mathcal{S}$
   - if $s_1, s_2 \in S$ then $s_1 s_2 \in S$

   Use structural induction to prove that no strings in $\mathcal{S}$ contain the substring yh. **Hint:** It may help to strengthen your induction hypothesis.

4. Define $A(n)$ as the smallest natural number containing exactly n substrings in its decimal representation which are prime numbers.
   For example, $A(2) = 13$, because the string '13' contains the prime numbers 3 and 13 itself (and is smaller than any other number with this property, such as 31). $A(6) = 373$, corresponding to the prime numbers 3 (which appears twice), 7, 37, 73, and 373.
   Prove that $A(n)$ is defined for each $n \in \mathbb{N}$. i.e. for each $n \in \mathbb{N}$, there exists a smallest natural number containing exactly n prime substrings.