CSC 2410F - Introduction to Graph Theory

Assignment #3 Due: Nov. 21, 2002

NB. Read the policy on "grading scheme" in the course outline. All work must be your own; reference all sources.

1. The following sequential algorithm is knows as *Sequential Colouring with Bichromatic Interchange*:

   Step 0: Order the vertices as $v_1, v_2, \cdots, v_n$ so that $d(v_i) \geq d(v_{i+1})$.

   Step 1: Assign colour 1 to v_1.

   Step 2: <u>for</u> i <u>from</u> 2 <u>to</u> n <u>do</u>
   Let G_i be the subgraph of G induced on $\{v_1, \cdots, v_i\}$. Also let $c_1, c_2, \cdots, c_p$ be the colours present in the neighbourhood of v_i in G_i. For any two such colours c_j and c_k, we let $C_{j,k}$ denote a connected component in the subgraph of G_i induced on the vertices that have either of these two colours.

   If there are colours c_j and c_k such that no connected component $C_{j,k}$ contains a neighbour of v_i that has been coloured j as well as a neighbour of v_i that has been coloured k, then we can remove the colour c_k from the neighbourhood of v_i by the following *interchange procedure*. For each component $C_{j,k}$ that contains a neighbour of v_i that has been coloured c_k interchange the colours c_j and c_k.

   Step 2.1: Repeat this interchange procedure until no other pair of colours can be interchanged.

   Step 2.2: Colour v_i with the smallest colour that is not present among the neighbours of v_i in G_i.

   (a) Show that the algorithm properly colours the vertices of a bipartite graph with at most two colours.

   (b) Consider the graph $H_p(V_p, E_p)$ where
   $V_p = \{x_i, y_i, z_i | 1 \leq i \leq p\}$
   $E_p = \{x_i y_j, x_i z_j, y_i z_j | i, j \leq p, i \neq j\}$.
   Use H_p to show that the above algorithm performs badly on non-bipartite graphs.

OVER

2. Let G be a graph with degree sequence $(d_1, d_2, \cdots, d_n)$ and let $\overline{G}$ have degree sequence $(d_1^{'}, d_2^{'}, \cdots, d_n^{'})$ where $d_1 \leq d_2 \leq \cdots \leq d_n$ and $d_1^{'} \leq d_2^{'} \leq \cdots \leq d_n^{'}$.

   (a) Show that if $d_j \geq d_j^{'}, \forall j \leq n/2$, then G has a Hamiltonian path.

   (b) Deduce that every self-complementary graph has a Hamiltonian path.

3. This question is from West's book (# 7.2.10). He defines a *trail* to be a walk in which no edge appears more than once. A *walk* is an alternating list of vertices and edges in a graph such that each vertex belongs to the edge before and the edge after it. A walk is *closed* if the last vertex is the same as the first. Note that, contrary to what's written in the Garey-Johnson book, the problem of determining whether a line graph is Hamiltonian is NP-complete.

   (a) Find a 2-connected non-Eulerian graph whose line graph is Hamiltonian.

   (b) Prove that $L(G)$ is Hamiltonian iff G has a closed trail that contains at least one endpoint of each edge.

4. Show that every graph with n vertices and m edges contains an independent set of size at least $\lceil \frac{n^2}{2m+n} \rceil$ and this bound is tight.

   Note that the best proof I know for this is algorithmic.

5. Given two perfect graphs $G(V, E)$ and $H(W, F)$ where each contains a complete subgraph on k vertices, K_G and K_H respectively. We form the *clique bonding* of G and H by identifying the vertices of K_G and K_H.

   (a) Show from first principles (i.e. just use the definition of perfect graphs) that the clique bonding of G and H is perfect.

   (b) Is there any other subgraph bonding that preserves perfection? I.e. either show that there exists a graph X (that is not a clique) such that X-bonding preserves perfection or show that for all X that is not a clique there is a pair of perfect graphs G and H such that the X-bonding of G and H is not perfect.