1. Let the following two expressions be given:

\[
\begin{align*}
  x_1 &\Rightarrow (x_2 \land (x_3 \lor x_4)) \quad (\text{exp}_1) \\
  (x_2 \lor \neg x_3) \land (x_1 \land x_4) &\quad (\text{exp}_2)
\end{align*}
\]

Let the order of variables \( x_1 < x_2 < x_3 < x_4 \) be given.

(a) Build BDDs for the two expressions, referring to them as \( BDD_1 \) and \( BDD_2 \).

(b) Compute Apply(\( \land \), \( BDD_1 \), \( BDD_2 \)). You may compare your answers with computing \( \text{exp}_1 \land \text{exp}_2 \) and building a BDD from it (this is not part of the assignment – it is for your benefit only).

(c) Compute Apply(\( \lor \), \( BDD_1 \), \( BDD_2 \)).
2. Prove the duality

$$\mu Z. f(Z) = \nu Z. \neg f(\neg Z)$$

Proof sketch: Assume that \( f : 2^S \to 2^S \) is a monotone function over subsets of a finite set \( S \). In this case, negation is set complement. Let \( g(Z) = \neg f(\neg Z) \), show by induction that \( g^i(S) = \neg f^i(\emptyset) \). The rest follows from the fact that fixpoint computation must converge after finitely many iterations.

3. Prove that \( AF \varphi = \mu Z. \varphi \lor AX Z \), i.e., prove

(a) \( \varphi \lor AX AF \varphi = AF \varphi \)

Need to show that for any state \( s \), \( \llbracket \varphi \lor AX AF \varphi \rrbracket(s) \Leftrightarrow \llbracket AF \varphi \rrbracket(s) \), which follows directly from the definition of \( AF \).

(b) \( \forall Y \cdot (Y = \varphi \lor AX Y) \Rightarrow (Y \supseteq AF \varphi) \)

Let \( F(Z) = \varphi \lor AX Z \). Note that if \( s \in \llbracket AF \varphi \rrbracket \) then there exists a bound \( k \) such that along every path from \( s \), a state in \( \llbracket \varphi \rrbracket \) is reached in at most \( k \) steps. Let \( s \) be such that \( s \in \llbracket AF \varphi \rrbracket \) and \( s \not\in Y \). Show by induction that \( s \in F^k(Y) = Y \).

4. Consider this model:

(a) Compute the transition relation \( R \) for this model.
\[ R = (x \land \neg y \land \neg x' \land \neg y') \lor (\neg x \land \neg y \land x' \land y') \lor \]
\[ (\neg x \land \neg y \land y' \land x') \lor (x \land y \land x' \land y') \]

(b) Symbolically, compute the value of \( AG EF y \). Check that the computation is correct by executing the explicit-state model-checking algorithm.

We make use of the following laws
\[
\exists x \cdot f \lor g = (\exists x \cdot f) \lor (\exists x \cdot g)
\]
\[
\exists x \cdot f \land x = f
\]

First step is to compute \( EF y = \mu Z \cdot y \lor EX Z \)

\[
EF_0 y = y \lor EX \bot
\]
\[
EF_1 y = y \lor \exists x', y' \cdot R \land y'
\]
\[
EF_2 y = y \lor \exists x', y' \cdot (\neg x \land \neg y \land x' \land y') \lor (x \land y \land x' \land y')
\]
\[
EF_3 y = y \lor (\neg x \land \neg y)
\]

Now we need to compute \( AG(EF y) = AG \top = \nu Z \land AX Z \)

\[
AG_0 \top = \top \land AX \top
\]
\[
= \top \land \forall x', y' \cdot \neg R \lor \top
\]
\[
= \top \land \forall x', y' \cdot \top
\]
\[
= \top
\]
(c) Symbolically, compute $EGy$.

We need to compute $EGy = \nu Z \cdot y \land EXZ$

\[
EG_{0y} \\
= y \land EX \top \\
= y \land \exists x', y' \cdot R \land \top \\
= y \land \exists x', y' \cdot R \\
= y \land (x \land y) \\
= y \land x
\]

\[
EG_{1y} \\
= y \land EX EG_{1y} \\
= y \land EX (x \land y) \\
= y \land \exists x', y' \cdot R \land (x' \land y') \\
= y \land ((\neg x \land \neg y) \lor (x \land y)) \\
= y \land x
\]