CSC 2125, Fall 2006
Homework 1 Partial Solution

1. Proof by induction on e.

   - Base case $e = x$. In this case, f is the identity function, which is clearly monotonic.
   - Base case $e = S'$. In this case, f is a constant function, which again is clearly monotonic.
   - Induction $e = e_1 \cup e_2$. Let $g(x) = e_1$ and $h(x) = e_2$. By induction, g and h are monotonic. So suppose $a \subseteq b$. Then $g(a) \subseteq g(b)$ and $h(a) \subseteq h(b)$. But then $g(a) \cup h(a) \subseteq g(b) \cup h(b)$. Since this holds for any a, b, $f = g \cup h$ is also monotonic.
   - Induction $e = e_1 \cap e_2$ is similar to above.

2. If arbitrary negation is included, then $f(x)$ may no longer be monotonic. For example, pick $f(x) = !x$ and $S = \{1\}$. Then $\emptyset \subset \{1\}$ but $f(\emptyset) = \{1\} \supset \emptyset$. On the other hand, $Out(stmt) = Gen(stmt) \cup (In(stmt) - Kill(stmt))$ is still monotonic. $Kill(stmt)$ is a fixed, constant set, thus its negation is as well, and so by problem 1 the Gen/Kill computation is monotonic.

3. (a) We need to show that $f \sqcup g$ always exists, which means we need to show that $(f \sqcup g)(x)$ exists for any $x \in A$. But $(f \sqcup g)(x) = (f(x)) \sqcup (g(x))$, which exists because f and g both have range A and A is a lattice. A similar argument holds for $\sqcap$. It would also be a good idea to show that $\leq'$ is a partial order, and again this follows naturally from the fact that $\leq$ is a partial order.

   (b) The height of $(A \to A, \leq')$ is hn. To see this, think of an element of the lattice as a tuple $(a_1, \ldots, a_n)$, where a_i is what the function maps the ith element of A to. Then $(a_1, \ldots, a_n) \leq' (b_1, \ldots, b_n)$ if $a_i \leq b_i$ for all i. The longest ascending chain in this lattice must go from $(\bot, \ldots, \bot)$ to $(\top, \ldots, \top)$, and this chain must have height hn, since each component of the tuple can increase exactly h times, and there are n of them.