Binary Decision Diagrams

- Representation of Boolean Functions
- BDDs, OBDDs, ROBDDs
- Operations
- Model-Checking over BDDs

Boolean Functions

Boolean functions: $\mathcal{B} = \{0, 1\}$,
$f : \mathcal{B} \times \cdots \times \mathcal{B} \to \mathcal{B}$

Boolean expressions:
$t ::= x \mid 0 \mid 1 \mid \neg t \mid t \wedge t \mid t \vee t \mid t \to t | t \leftarrow t$

Truth assignments: ρ,
$[v_1/x_1, v_2/x_2, \cdots, v_n/x_n]$

Satisfiable: Exists ρ s.t. $t[\rho] = 1$

Tautology: Forall ρ, $t[\rho] = 1$

Truth Tables

xyz	$x \rightarrow y, z$
000	0
001	1
010	0
011	1
100	0
101	0
110	1
111	1

$x_1 \cdots x_n$	$f(x_1, \cdots x_n)$
$0 \cdots 0$	1
$0 \cdots 1$	0
$\vdots$	$\vdots$
$1 \cdots 1$	0

2^n entries

What is a good representation of boolean functions?

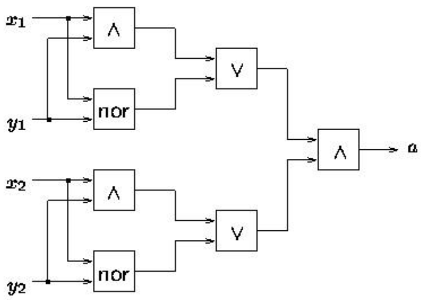
Perfect representation is hopeless:

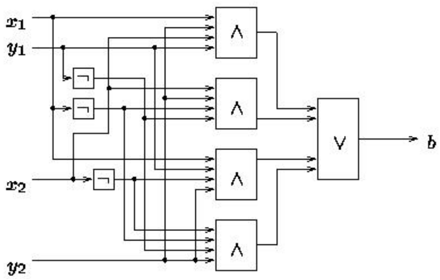
Theorem 1 (Cook's Theorem)
Satisfiability of Boolean expressions is NP-complete.

(Tautology-checking is co-NP-complete)

Good representations are
  compact and
  efficient
on real-life examples

Combinatorial circuits





Are these equivalent? Do they represent a tautology? Are they satisfiable?

Shannon Expansion

Def: $x \to y_0, y_1 = (x \wedge y_0) \vee (\neg x \wedge y_1)$

x is the test expression and thus this is an if-then-else.

We can represent all operators using if-then-else on unnegated variables and constants 0(false) and 1(true). This is called INF.

Shannon expansion w.r.t. x:

$$t = x \to t[1/x], t[0/x]$$

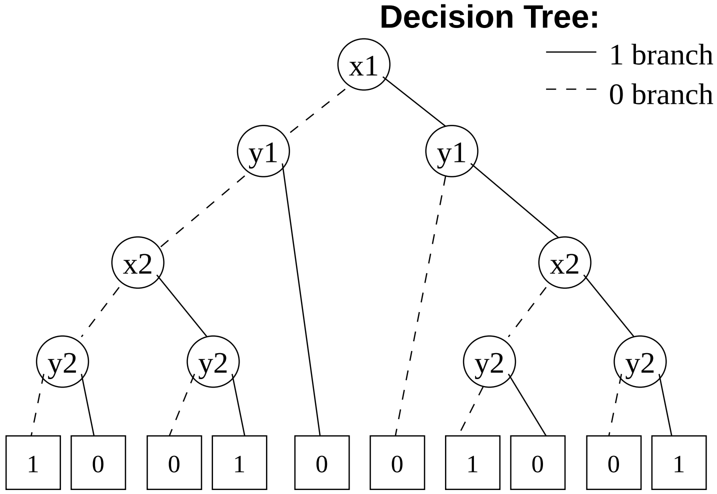
Any boolean expression is equivalent to an expression in INF.

Example

$t = (x_1 \Leftrightarrow y_1) \wedge (x_2 \Leftrightarrow y_2)$. Represent this in INF form with order x_1, y_1, x_2, y_2.

$t = x_1 \rightarrow t_1, t_0$
$t_0 = y_1 \rightarrow 0, t_{00}$
(since $x_1 = 1, y_1 = 0 \rightarrow t = 0$)
$t_1 = y_1 \rightarrow t_{11}, 0$
(since $x_1 = 0, y_1 = 1 \rightarrow t = 0$)
$t_{00} = x_2 \rightarrow t_{001}, t_{000}$
$t_{11} = x_2 \rightarrow t_{111}, t_{000}$
$t_{000} = y_2 \rightarrow 0, 1$ ($x_1 = 0, y_1 = 0, x_2 = 0$)
$t_{001} = y_2 \rightarrow 1, 0$ ($x_1 = 0, y_1 = 0, x_2 = 1$)
$t_{110} = y_2 \rightarrow 0, 1$ ($x_1 = 1, y_1 = 1, x_2 = 0$)
$t_{111} = y_2 \rightarrow 1, 0$ ($x_1 = 1, y_1 = 1, x_2 = 1$)

Decision Tree:

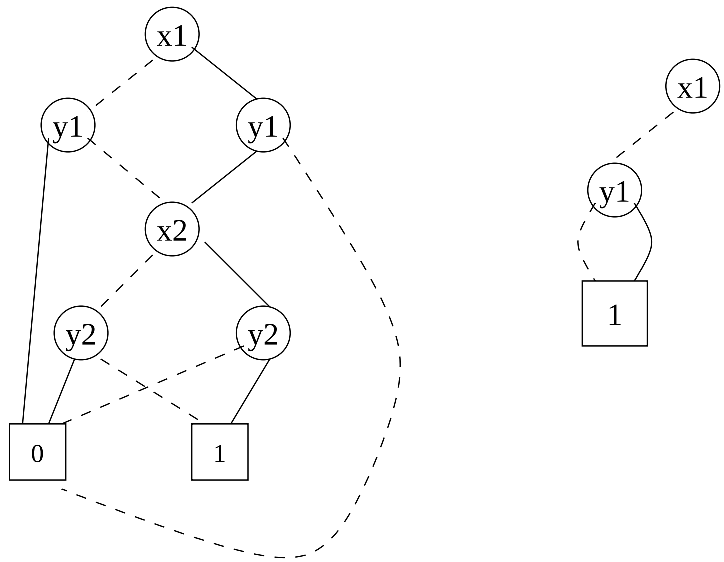


Lots of common subexpressions:
- identify them!

BDDs – directed acyclic graph of Boolean expressions. If the variables occur in the same ordering on all paths from root to leaves, we call this OBDD.

Example OBDD

OBDD for $(x_1 \Leftrightarrow y_1) \wedge (x_2 \Leftrightarrow y_2)$ with ordering $x_1 < y_1 < x_2 < y_2$



If an OBDD does not contain any redundant tests, it is called ROBDD.

ROBDDs

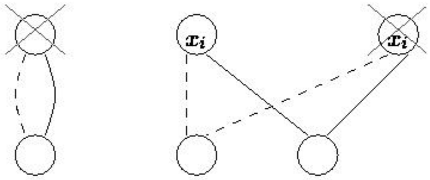
A *Binary Decision Diagram* is a rooted, directed, acyclic graph (V,E). V contains (up to) two terminal vertices, $0,1 \in V$. $v \in V \setminus \{0,1\}$ are non-terminal and have attributes *var*(v), and *low*(v), *high*(v) $\in V$.

A BDD is *ordered* if on all paths from the root the variables respect a given total order.

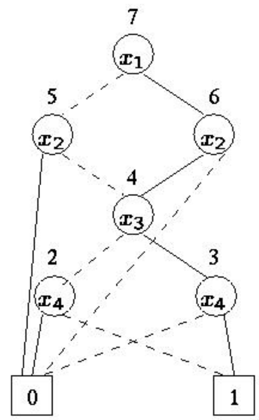
A BDD is *reduced* if for all non-terminal vertices u,v,

1) *low*(u) $\neq$ *high*(u)
2) *low*(u) = *low*(v), *high*(u) = *high*(v), *var*(u) = *var*(v) implies $u = v$.

ROBDD Examples



reducedness



$(x_1 \Leftrightarrow x_2) \land (x_3 \Leftrightarrow x_4)$

Canonicity of ROBDDs

Lemma 1 (Canonicity lemma) For any function $f : \mathcal{B}^n \to \mathcal{B}$ there is exactly one ROBDD b with variables $x_1 < x_2 < \cdots < x_n$ such that

$t_b[v_1/x_1, \cdots, v_n/x_n] = f(v_1, \cdots, v_n)$

for all $(v_1, ..., v_n) \in \mathcal{B}^n$.

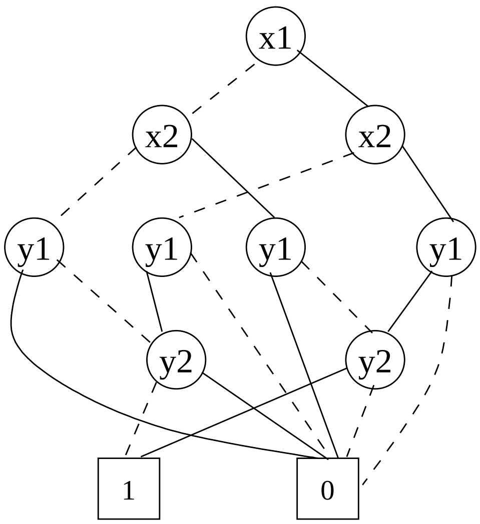
Consequences:

- b is a tautology if and only if $b = \boxed{1}$
- b is satisfiable if and only if $b \neq \boxed{0}$

But...

The size of ROBDD depends *significantly* on the chosen variable ordering!

Example: ROBDD for $(x_1 \Leftrightarrow y_1) \wedge (x_2 \Leftrightarrow y_2)$ with ordering $x_1 < x_2 < y_1 < y_2$



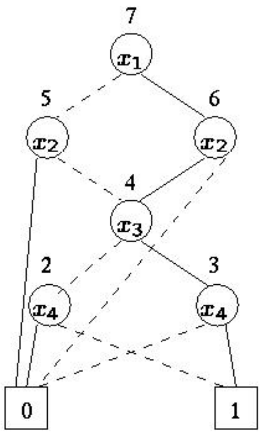
Under ordering $x_1 < y_1 < x_2 < y_2$ had 6 nodes.

Furthermore...

- The size according to one ordering may be exponentially smaller than another ordering.
- Figuring out the optimal ordering of variables is co-NP-complete.
- Some functions have small size independent of ordering, e.g. parity.
- Some functions have large size independent of ordering, e.g., multiplication

Implementing BDDs

{root: integer; var, low, high: array of integer;}



	var	*low*	*high*
0	?	?	?
1	?	?	?
2	4	1	0
3	4	0	1
4	3	2	3
5	2	4	0
6	2	0	4
7	1	5	6

Helper Functions: Makenode and Hashing

Makenode ensures reducedness using a hash table

$H : (i, l, h) \rightarrow u$

Makenode(H, *max*, b, i, l, h)

if $l = h$ **then return** l
else if *member*(H, i, l, h)
  then return *lookup*(H, i, l, h)
else *max* $\leftarrow$ *max* + 1
  b.*var*(*max*) $\leftarrow i$
  b.*low*(*max*) $\leftarrow l$
  b.*high*(*max*) $\leftarrow h$
  insert(H, i, l, h, *max*)
  return *max*

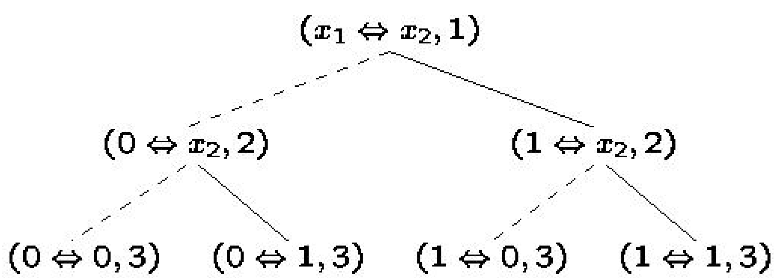
Build

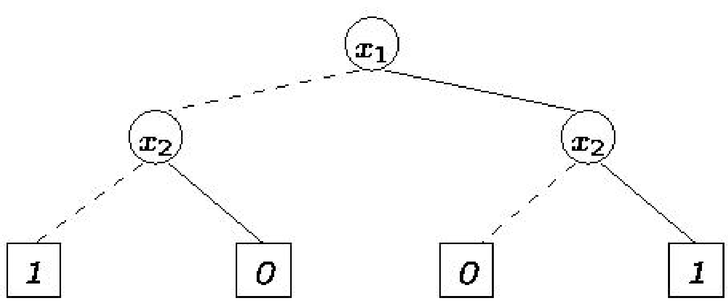
Build: Maps a Boolean expression t into an ROBDD.

function *Build*(t)
    $H \leftarrow$ *emptytable*; *max* $\leftarrow 1$
    b.*root* $\leftarrow$ *build'*(t, 1)
    return b

function *build'*(t, i)
    if $i > n$ **then**
      if t is false **then return** 0
        else return 1
      else $l \leftarrow$ *build'*($t[0/x_i], i+1$)
        $h \leftarrow$ *build'*($t[1/x_i], i+1$)
        return *makenode*(H, *max*, b, i, l, h)

Build Example

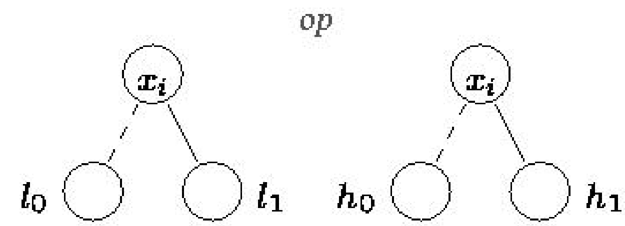


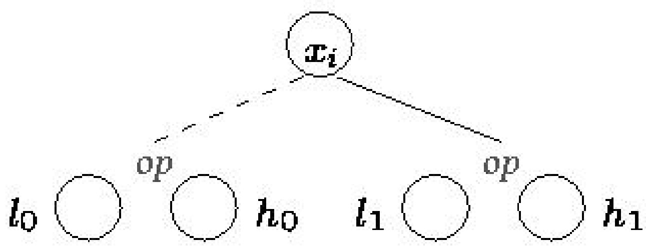


Boolean Operations on ROBDDs

Ordering: $x_1 < \cdots < x_n$

$(x_i \to l_1, l_0)\ op\ (x_i \to h_1, h_0) = x_i \to (l_1\ op\ h_1), (l_0\ op\ h_0)$



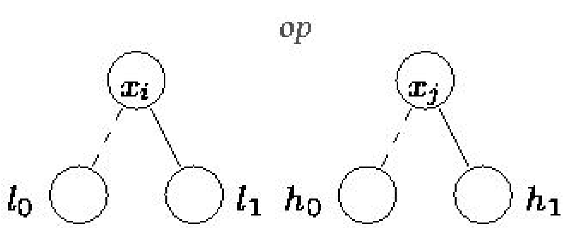


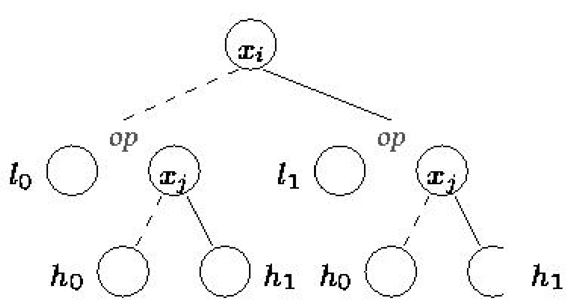
Boolean Operations on ROBDDs(Cont'd)

If $x_i < x_j$:

$(x_i \to l_1, l_0)\ op\ (x_j \to h_1, h_0) =$

$x_i \to (l_1\ op\ (x_j \to h_1, h_0)), (l_0\ op\ (x_j \to h_1, h_0))$

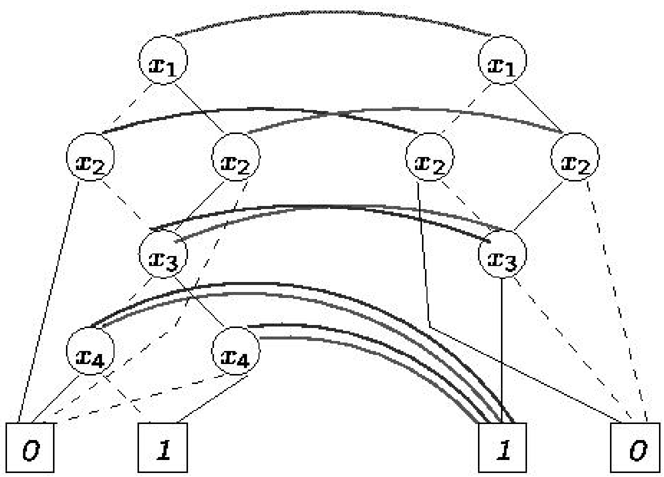




Function *Apply*

Used to perform operations on two ROBDDs.

Example:



Can be either recursive or using dynamic programming.

Other Operations on ROBDDs

Restrict – $b[v/x]$”
given a truth assignment for x, compute ROBDD for b

Size – *size*(b) = $|\{\rho \mid b[\rho] = 1\}|$
”number of valid truth assignments”

Anysat – *anysat*(b) = ρ, for some ρ with $b[\rho] = 1$
”give a satisfying assignment”

Compose – *compose*$(b, x, b') = b[x/b']$
”substitute b' for all free occurrences of x”

Existential quantification – $\exists x.b = b[x/0] \vee b[x/1]$

Using dynamic hash-table implementation, can get amortized cost for operations to be $O(1)$.

Representing Boolean Functions

Representation of boolean functions	compact?	satisf'ty	validity
Prop. formulas	often	hard	hard
Formulas in DNF	sometimes	easy	hard
Formulas in CNF	sometimes	hard	easy
Ordered truth tables	never	hard	hard
Reduced OBDDs	often	easy	easy

Representation of boolean functions	$\wedge$	$\vee$	$\neg$
Prop. formulas	easy	easy	easy
Formulas in DNF	hard	easy	hard
Formulas in CNF	easy	hard	hard
Ordered truth tables	hard	hard	hard
Reduced OBDDs	medium	medium	easy

Uses of ROBDDs

Symbolic reasoning about:

- Combinatorial circuits
- Sequential circuits
- Automata
- Program analysis (theorem-proving)

and

- Temporal logic model checking