Faculty of Arts & Science / Physical Sciences Division
University of Toronto
CSC148/A58—Introduction to Computer Science, Fall 2000

Assignment 4: Proofs

Due: Week 9

This assignment involves no programming; all questions require pencil-and-paper solutions only.

1. Prove by induction that for all $n \geq 1$,

   $(2 \times 1 - 1) + (2 \times 2 - 1) + (2 \times 3 - 1) + \ldots + (2 \times n - 1) = n^2$.

2. Consider the assertion:

   "The number $n^2 + 5n + 1$ is even for all $n \geq 1$."

   (a) Below is the outline of the induction step for a proof of this assertion. Complete it.

   Let k be any integer ≥ 1.

   Induction Hypothesis: Assume $k^2 + 5k + 1$ is even.

   Induction Step: Prove that $(k + 1)^2 + 5(k + 1) + 2$ must also be even.

   Body of the induction step missing.

   (b) The assertion above is actually wrong. In fact, $n^2 + 5n + 1$ is *odd* for all $n \geq 1$. Explain how we can do the proof in (a) for an assertion that is wrong.

   (c) Use mathematical induction to show that $n^2 + 5n + 1$ is odd for all $n \geq 1$.

3. Consider attending a party where everyone shakes hands with everyone else. For example, at a three-person party attended by Sara, Tom and Bill, there will be 3 handshakes:

   Sara with Tom
   Sara with Bill
   Bill with Tom

   Use mathematical induction to prove that the total number of handshakes made in a party attended by n people ($n \geq 1$) is $n(n - 1)/2$.

4. Consider the function F, defined by the following recurrence relation:

   $F_1 = 1$

   $F_2 = 1$

   $F_n = F_{n-1} + F_{n-2}$, for $n \geq 2$.

   So, $F_3 = F_1 + F_2 = 1 + 1 = 2$, etc. The values of F, namely, $F_1, F_2, F_3, F_4, F_5, \ldots$, which are 1, 1, 2, 3, 5, ..., are called the *Fibonacci numbers*, after the Italian mathematician Leonardo Fibonacci, who first studied their properties in the year 1202.

   Prove that the sequence $F_3, F_6, F_9, \ldots$ consists of even numbers only.

5. An advertisement for a tennis magazine says:

   (a) "If I am not playing tennis, I am watching tennis".

   (b) "If I am not watching tennis, I am reading about tennis".

   Presumably, each person can do only one thing at a time, so to this list we add

   (c) Noone can read about tennis and play tennis at the same time.

   (d) Noone can watch tennis and read about tennis at the same time.

   (e) Noone can watch tennis and play tennis at the same time

   Prove that the author of this ad is not reading about tennis.

6. The princess had two caskets, one gold and one silver. Into one she placed her portrait and into the other – a dagger. On the gold casket she wrote the inscription: "The portrait is not here". On the silver casket she wrote: "Exactly one of these inscriptions is true". She explained to the suitor that each inscription is either true or false (not both), but on the basis of the inscriptions he must choose the casket. If he chooses the one with the portrait, he can marry her; if he chooses the one with the dagger, he must kill himself.

   Prove that the portrait is in the golden casket.

What to Hand In

It is in your best interest to submit legibly written (or better, typed) proofs, organized clearly and described precisely. That way the grader will be able to follow your reasoning. For proofs by induction, state precisely what is the statement to be proved, and separate your proof into the three distinct steps labeled **Base Case**, **Induction Hypothesis** and **Induction Step**. You also need to explicitely introduce the variable you are using in the induction step. Do not use a strong induction hypothesis when weak induction is sufficient.

Submit your work on paper only; there is no electronic submission for this assignment. Staple the cover sheet securely to your work. An envelope is not required, and in fact, we prefer that you not use one for this assignment.