Vertex Cover Resists SDPs Tightened by Local Hypermetric Inequalities

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Abstract. We consider the standard semidefinite programming (SDP) relaxation for VERTEX COVER to which all hypermetric inequalities supported on at most k vertices have been added. We show that the integrality gap for such SDPs remains $2 - o(1)$ as long as $k = O(\sqrt{\log n / \log\log n})$. This extends successive results by Kleinberg-Goemans, Charikar and Hatami et al. which analyzed integrality gaps of the standard VERTEX COVER SDP relaxation as well as for SDPs tightened using triangle and pentagonal inequalities.

Our result is complementary but incomparable to a recent result by Georgiou et al. proving integrality gaps for VERTEX COVER SDPs in the Lovász-Schrijver hierarchy. One of our contributions is making explicit the difference between the SDPs considered by Georgiou et al. and those analyzed in the current paper. We do this by showing that VERTEX COVER SDPs in the Lovász-Schrijver hierarchy fail to satisfy any hypermetric constraints supported on independent sets of the input graph.

1 Introduction

A *vertex cover* for a graph is a subset of vertices that touches all edges in the graph. Determining the approximability of the minimum VERTEX COVER problem on graphs is one of the outstanding problems in theoretical computer science. While there exists a trivial 2-approximation algorithm, considerable efforts have failed to obtain an approximation ratio better than $2 - o(1)$. On the other hand, the strongest PCP-based hardness result known [8] only shows that 1.36-approximation of VERTEX COVER is NP-hard. Only by assuming Khot's Unique Game Conjecture [15] can it be shown that $2-o(1)$-approximation is NP-hard [16].

Several recent papers [17, 12, 4, 14, 13, 10] examine whether semidefinite programming (SDP) relaxations of VERTEX COVER might yield better approximations. Goemans and Williamson [11] introduced semidefinite programming relaxations as an algorithmic technique using it to obtain a 0.878-approximation for MAX-CUT. Since then semidefinite programming has arguably become our most powerful tool for designing approximation algorithms. Indeed, for many NP-hard optimization problems the best approximation ratios are achieved using SDP-based algorithms.

Given a graph $G = (V, E)$, the standard SDP relaxation for VERTEX COVER is

$$\begin{array}{lll} \min & \sum_{i\in V}(1+\mathbf{v}_0\cdot\mathbf{v}_i)/2 & \\ \text{s.t.} & (\mathbf{v}_0-\mathbf{v}_i)\cdot(\mathbf{v}_0-\mathbf{v}_j)=0 & \forall ij\in E \\ & \|\mathbf{v}_i\|=1 & \forall i\in\{0\}\cup V \end{array} \tag{1}$$

Halperin [12] employed this relaxation together with an appropriate rounding technique to obtain a $(2 - \Omega(\log\log\Delta/\log\Delta))$-approximation for VERTEX COVER for graphs with maximal degree Δ. Unfortunately, Kleinberg and Goemans [17] showed that in general this relaxation has an integrality gap of $2 - o(1)$.

One possible avenue for decreasing this integrality gap comes from the following simple observation: for any integral (or rather, one-dimensional) solution, $\|\mathbf{v}_i - \mathbf{v}_j\|^2$ is an ℓ_1 metric. Therefore the addition of inequalities on the distances $\|\mathbf{v}_i - \mathbf{v}_j\|^2$ that are valid for ℓ_1 metrics may yield a possible tightening of the SDP (note that the constraint $(\mathbf{v}_0 - \mathbf{v}_i)\cdot(\mathbf{v}_0 - \mathbf{v}_j) = 0$ in SDP (1) is in fact the following distance constraint "in disguise": $\|\mathbf{v}_i - \mathbf{v}_0\|^2 + \|\mathbf{v}_j - \mathbf{v}_0\|^2 = \|\mathbf{v}_i - \mathbf{v}_j\|^2$).

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For example, since ℓ_1 metrics satisfy the triangle inequality, we could add the following constraint to SDP (1):

$$\|\mathbf{v}_i - \mathbf{v}_j\|^2 + \|\mathbf{v}_j - \mathbf{v}_k\|^2 \geq \|\mathbf{v}_i - \mathbf{v}_k\|^2 \qquad \forall i, j, k \in \{0\} \cup V \ . \tag{2}$$

This ℓ_2^2 triangle inequality plays a crucial role in the breakthrough Arora-Rao-Vazirani SPARSEST CUT algorithm [2]. This suggests that the addition of inequalities satisfied by ℓ_1 metrics may be exactly what is needed to get a $2 - \Omega(1)$ approximation for VERTEX COVER.

Indeed, Hatami et al. [13] prove that if SDP (1) is strengthened by requiring that the distances $\|\mathbf{v}_i - \mathbf{v}_j\|^2$ satisfy *all* ℓ_1 inequalities (i.e., the vectors $\mathbf{v}_i$ equipped with the ℓ_2^2 norm $\|\cdot\|^2$ are ℓ_1-embeddable), then the resulting relaxation has no integrality gap. Of course, the caveat here is that the resulting relaxation has exponentially many constraints and is hence intractable. To obtain a tractable relaxation (or at least one computable in subexponential time), our relaxation must use only a limited subset of ℓ_1 inequalities.

One canonical subclass of ℓ_1 inequalities is the discrete and easily-described class of *hypermetric* inequalities (see the Preliminaries for definitions). These include the triangle inequalities as well as the so-called pentagonal, heptagonal, etc., inequalities. That such inequalities might be useful for designing improved approximation algorithms is illustrated, for example, in a work by Avis and Umemoto [3]. Avis and Umemoto show that for dense graphs, linear programming relaxations of MAX CUT based on the k-gonal inequalities have integrality gap at most $1 + 1/k$. This, in a sense, gives rise to an LP-based PTAS for MAX CUT.

Unfortunately, for VERTEX COVER Charikar [4] showed that even with the addition of the triangle inequality (2) the integrality gap of SDP (1) remains $2 - o(1)$. However, Karakostas [14] did show that adding the triangle inequality (as well as the "antipodal" triangle inequalities $(\pm\mathbf{v}_i - \pm\mathbf{v}_j) \cdot (\pm\mathbf{v}_i - \pm\mathbf{v}_k) \geq 0$) yields a $(2 - \Omega(1/\sqrt{\log n}))$-approximation for VERTEX COVER, currently the best ratio achievable by any algorithm. Hatami et al. [13] subsequently showed that Karakostas's SDP even with the addition of the pentagonal inequalities has integrality gap $2 - o(\sqrt{\log \log n / \log n})$.

In this work we rule out the possibility that adding local hypermetric constraints improves the integrality gap of VERTEX COVER SDPs:

Theorem 1. *The tightening of the standard SDP for* VERTEX COVER *with all hypermetrics that are supported on $O(\sqrt{\log n / \log \log n})$ points has integrality gap $2 - o(1)$.*

As mentioned above, Hatami et al. [13] show that adding the constraint that solutions to SDP (1) be ℓ_1-embeddable results in an SDP with no integrality gap. Theorem 1 then immediately gives the following corollary about ℓ_2^2 metrics:

Corollary 1. *There exist ℓ_2^2 metrics that are not isometrically embeddable into ℓ_1, yet satisfy all hypermetric inequalities supported on $O(\sqrt{\log n / \log \log n})$ points.*

It is interesting to compare Corollary 1 with recent results contrasting local and global phenomena in metric spaces. In [1, 5] the authors describe metric spaces that cannot be well-embedded into ℓ_1 but locally every small subset embeds isometrically into ℓ_1. In contrast, our corollary shows the existence of a metric that locally resembles ℓ_1 (although not provably ℓ_1) but globally does not embed isometrically into ℓ_1. From this standpoint, this is far weaker than [1, 5]. However, the metric we supply is also an ℓ_2^2 metric. Finding ℓ_2^2 metrics that are far from being ℓ_1 proved to be a very challenging task (see Khot and Vishnoi's work [6] motivated by integrality gap instances for SPARSEST CUT). To the best of our knowledge, there are no known results that point to such metrics which further satisfy any local conditions beyond the obvious triangle inequality.

A result related to Theorem 1 was proved by Georgiou et al. in [10]. The main result of that paper showed that SDP relaxations obtained by tightening the standard linear programming relaxation for VERTEX COVER using $O(\sqrt{\log n / \log \log n})$ rounds of the LS_+ "lift-and-project" method of Lovász and Schrijver [18] have integrality gap $2 - o(1)$. The SDPs considered in [10] seem intimately related to those obtained by adding local ℓ_1 or hypermetric constraints. Indeed, it is well known that relaxations from the LP Lovász-Schrijver hierarchy satisfy all valid local LP constraints. However, it is also known [10] that relaxations from the SDP Lovász-Schrijver hierarchy do not necessarily satisfy all valid local SDP constraints. In particular, the

VERTEX COVER SDP relaxation obtained after k rounds of the LS_+ method is not obviously comparable to the relaxation obtained by adding all order-k hypermetric inequalities to SDP (1). In section 4 we show in a strong sense the incomparability of these relaxations: Fix any subset S of vertices that is an independent set in the underlying graph. We then find a hypermetric inequality supported on all points of S that is nevertheless not valid for any VERTEX COVER SDP in the Lovász Schrijver hierarchy. In particular, this shows that the integrality gaps proved in [10] do not preclude the possibility that adding such concrete constraints as, say, the "heptagonal" inequalities, may result in an improved SDP relaxation.

We briefly describe how we prove Theorem 1. We use the same graph family as in [17, 4, 13, 10]. The SDP solution can be thought of as an ℓ_1 metric to which a small perturbation is applied. This perturbation is characterized by two "infinitesimal" parameters, γ and ϵ, relating to the graph and the integrality gap, respectively. We show that hypermetric inequalities that are supported on $k \geq 4$ points, one of which is $\mathbf{v}_0$, must have a slack component that depends on k and on ϵ and γ, that will be maintained as long as $k\gamma = O(\epsilon)$. The case when $k = 3$ (i.e., the triangle inequality) is covered by [4] and [13], and the case where $\mathbf{v}_0$ does not participate in the inequality is handled by the fact that the metric formed by the remaining vectors is an ℓ_1 metric. Setting ϵ to an arbitrary small constant, and setting γ to $\Theta(\sqrt{\log\log n/\log n})$ provides the bound in our theorem.

2 Preliminaries

Given two vectors $\mathbf{x}, \mathbf{y} \in \{-1, 1\}^n$ their *Hamming distance* $d_H(\mathbf{x}, \mathbf{y})$ is $|\{i \in [n] : x_i \neq y_i\}|$. For two vectors $\mathbf{u} \in \mathbb{R}^n$ and $\mathbf{v} \in \mathbb{R}^m$ denote by $(\mathbf{u}, \mathbf{v}) \in \mathbb{R}^{n+m}$ the vector whose projection on the first n coordinates is $\mathbf{u}$ and on the last m coordinates is $\mathbf{v}$.

The tensor product $\mathbf{u} \otimes \mathbf{v}$ of vectors $\mathbf{u} \in \mathbb{R}^n$ and $\mathbf{v} \in \mathbb{R}^m$ is the vector in $\mathbb{R}^{nm}$ indexed by ordered pairs from $n \times m$ and that assumes the value $\mathbf{u}_i\mathbf{v}_j$ at coordinate (i, j). Define $\mathbf{u}^{\otimes d}$ to be the vector in $\mathbb{R}^{n^d}$ obtained by tensoring $\mathbf{u}$ with itself d times. Let $P(x) = c_1x^{t_1} + \ldots + c_qx^{t_q}$ be a polynomial with nonnegative coefficients. Then T_P is the function that maps a vector $\mathbf{u}$ to the vector $T_P(\mathbf{u}) = (\sqrt{c_1}u^{\otimes t_1}, \ldots, \sqrt{c_q}u^{\otimes t_q})$.

Fact: For all vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$, $T_P(\mathbf{u}) \cdot T_P(\mathbf{v}) = P(\mathbf{u} \cdot \mathbf{v})$.

Metrics and ℓ_1 Inequalities We quickly review the facts we need about ℓ_1 inequalities. Deza and Laurent [7] is a good source for more information.

A finite metric space is an ℓ_1 *metric* if it can be embedded in ℓ_1-normed space so that all distances remain unchanged. It is easy to see that the set $\mathcal{C}$ of all ℓ_1 metrics on a fixed number of points is a convex cone. Let X be a set of size n. A subset S of X is associated with a metric $\delta_S(x, y)$ that is called a *cut metric* and is defined as $|\chi_S(x) - \chi_S(y)|$, where $\chi_S(\cdot)$ is the characteristic function of S. These metrics are the extreme rays of $\mathcal{C}$; namely, every ℓ_1 metric is a positive linear combination of cut metrics. This fact leads to a simple characterization of all inequalities that are valid for ℓ_1 metrics as follows. Consider the polar cone of $\mathcal{C}$,

$$\mathcal{C}^* = \{B \in \mathbb{R}^{n\times n} | B \cdot D \leq 0 \text{ for all } D \in \mathcal{C}\},$$

where by $B \cdot D$ we denote the matrix inner product of B and D, that is $B \cdot D = \text{trace}(BD^t) = \sum_{i,j} B_{ij}D_{ij}$. Notice that for B to be in $\mathcal{C}^*$ it is enough to require that $B \cdot \delta_S \leq 0$ for all cuts S. By definition it is clear that any $B \in \mathcal{C}^*$ defines a valid inequality such that $\sum_{i,j} B_{ij}d_{ij} \leq 0$ whenever d is an ℓ_1 metric. Conversely, (strong) duality implies that if d satisfies all inequalities of this type for every $B \in \mathcal{C}^*$ then d is an ℓ_1 metric.

A special canonical class of ℓ_1 inequalities is the class of *hypermetric inequalities*. Let $\mathbf{b} \in \mathbb{Z}^k$, with $\sum_{i=1}^k b_i = 1$. It can be easily verified that $B = \mathbf{b}\mathbf{b}^t$ is in $\mathcal{C}^*$. The inequality $\sum_{i,j} b_ib_jd_{ij} \leq 0$ is called a *hypermetric*. If we further require $\mathbf{b} \in \{-1, 1\}^k$, in which case the hypermetric is called *pure*, we obtain the k-gonal inequalities, e.g., the triangle inequality for $k = 3$, pentagonal inequality for $k = 5$, etc.

3 Construction and Proof

Fix arbitrarily small constants $\gamma, \epsilon > 0$ such that $\epsilon > 3\gamma$, and let m be a sufficiently large integer. The *Frankl-Rödl graph* G_m^γ is the graph with vertices $\{-1, 1\}^m$ and where two vertices $i, j \in \{-1, 1\}^m$ are adjacent if

$d_H(i,j) = (1-\gamma)m$. A classical result of Frankl and Rödl [9] implies that the size of a minimal vertex cover in G_m^γ is $2^m(1-o(1))$ whenever $\gamma = \Omega(\sqrt{\log m/m})$. We denote the vertices V of G as vectors $\mathbf{w}_i \in \{-1,1\}^m$ (the association of index i with a vector in the cube is arbitrary) and normalize these to get unit vectors $\mathbf{u}_i = \frac{1}{\sqrt{m}}\mathbf{w}_i$.

Consider the polynomial

$$P(x) = \beta x(x+1)^{\frac{2m}{\gamma}} + \alpha x^{\frac{1}{\gamma}} + (1-\alpha-2\beta)x \ ,$$

where the constants $\alpha, \beta > 0$ will be defined below. Let $\mathbf{z}_0 = (1,0\ldots,0)$, $\mathbf{z}_i = (2\epsilon, \sqrt{1-4\epsilon^2}T_P(\mathbf{u}_i))$, where $T_P(\mathbf{v})$ is the tensoring of $\mathbf{v}$ induced by the polynomial P. We fix the values of α and β defining P (and hence, defining the vectors $\mathbf{z}_i$) according to the following lemma implicit in [10]:

Lemma 1 ([10]). *Suppose $\frac{2m}{\gamma}$ and $\frac{1}{\gamma}$ are even and that m is significantly larger than $1/\gamma$. Suppose further that $\epsilon > 3\gamma$. Then there exist constants $\alpha, \beta > 0$ satisfying*

$$\alpha < 7.5\gamma \ ,$$
$$2\beta + \alpha > \frac{4\epsilon}{1+2\epsilon} - 4\gamma \ ,$$

such that the vectors $\mathbf{z}_0, \mathbf{z}_1, \ldots, \mathbf{z}_n$ satisfy both the standard VERTEX COVER *SDP* (1) *and the triangle inequality 2.*

A translated version of the vector set $\{\mathbf{z}_0, \mathbf{z}_1, \ldots, \mathbf{z}_n\}$ lay at root of the LS_+ lower bounds proved in [10]. Specifically, the Gram matrix of the vectors $\mathbf{v}_i = \frac{\mathbf{z}_0+\mathbf{z}_i}{2}$ was shown to be a solution for the VERTEX COVER SDP resulting from $O(\sqrt{\log n/\log\log n})$ rounds of LS_+ lift-and-project.

The remainder of this section is devoted to proving the following theorem.

Theorem 2. *The vectors $\mathbf{z}_0, \mathbf{z}_1, \ldots, \mathbf{z}_n$ satisfy all hypermetric inequalities on r points, $r \leq \frac{2}{45}\frac{\epsilon}{\gamma}$.*

We claim that Theorem 1 follows immediately from Theorem 2. Indeed, note first that the value of SDP (1) on the vectors $\mathbf{z}_0, \mathbf{z}_1, \ldots, \mathbf{z}_n$ is $(1+\epsilon)2^{m-1}$. On the other hand, recall that the underlying graph G_m^γ has minimal vertex cover size $(1-o(1))2^m$ whenever $\gamma = \Omega(\sqrt{\log m/m})$. Hence, Theorem 1 follows by taking $\epsilon > 0$ to be any arbitrarily small constant and $\gamma = \Omega(\sqrt{\log m/m})$.

As an aside, we note that our vectors $\{\mathbf{z}_i\}$ also satisfy the "antipodal" triangle inequalities $(\pm\mathbf{z}_i - \pm\mathbf{z}_j)\cdot(\pm\mathbf{z}_i - \pm\mathbf{z}_k) \geq 0$ for all $i,j,k \in \{0\} \cup V$. Recall that these inequalities define the SDP at root of Karakostas's [14] VERTEX COVER algorithm. That our vectors satisfy these inequalities can be seen as follows. Consider the subset $\{\mathbf{z}_i\}_{i\geq 1}$. For each coordinate, the vectors in this subset take on at most 2 different values, and hence this subset is ℓ_1-embeddable. Moreover, this remains true even if we replace some (or all) of the $\mathbf{z}_i$ by $-\mathbf{z}_i$. Hence, it suffices to consider only the "antipodal" triangle inequalities involving $\mathbf{z}_0$. The validity of these inequalities then follows easily from the fact that the $\mathbf{z}_i$ satisfy the standard triangle inequalities (by Lemma 1) and the fact that the value of $\mathbf{z}_i \cdot \mathbf{z}_0$ does not depend on i.

Before giving the proof of Theorem 2 we give some intuition. Note that the vector set $\{\mathbf{z}_i\}$ is the result of a perturbation applied to the following simple-minded ℓ_1 metric: Let $D = \{\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be the metric obtained by taking $\mathbf{v}_i$ to be the (normalized version of) the vectors of the m-dimensional cube, and let $\mathbf{v}_0$ be a unit vector perpendicular to all $\mathbf{v}_i$. Notice that these vectors are precisely the vectors we would have obtained if we had used the polynomial $P(x) = x$ to define the tensored vectors $\mathbf{z}_i$ (corresponding to taking $\epsilon = \gamma = 0$). The metric D is easily seen to be an ℓ_1 metric: take the Hamming cube and place the zeroth point at the origin to get an ℓ_1 embedding that is an isometry. Since D is ℓ_1, every hypermetric inequality is valid for it. On the other hand, D does not satisfy even the basic conditions of SDP (1) (e.g., the edge constraints) with respect to our graph of interest (i.e., G_m^γ with $\gamma > 0$) and any basic attempts to remedy that will violate even the triangle inequality: A subtle way of perturbing D via the tensoring polynomial P will be required. By focusing on the pure hypermetrics, we can give some intuition about why our construction works and why the critical value of k is $O(\epsilon/\gamma)$ (for non-pure hypermetrics, this intuition is less accurate). Given any choice of $\alpha, \beta > 0$ we get a set of tensored vectors $\mathbf{z}_i$ whose distances are a perturbation of D by an additive factor D_Δ. As mentioned in the proof outline in the introduction and in light of Lemma 1, it suffices to restrict our attention only to inequalities supported on more than three points. Since any given pure hypermetric inequality defined by the b_i's must be satisfied by D, it is sufficient to prove that it is satisfied for the perturbed component of the metric, i.e., D_Δ. Analyzing this inequality on D_Δ then shows that $\sum_{i,j} b_i b_j d_{ij} \le -2\epsilon + C\gamma k$, where C is a universal constant and the d_{ij} are the distances defined by D_Δ. Consequently, as long as $k = O(\epsilon/\gamma)$, the inequality holds for D_Δ. Hence it holds for $D + D_\Delta$, the metric resulting from the $\mathbf{z}_i$'s as well.

Proof (of Theorem 2). By Lemma 1 we already know that the vectors satisfy all hypermetric inequalities on three points, namely, the triangle inequalities.

So we only need to show that the solution satisfies hypermetric inequalities on 4 or more points. This is an important point since the arguments we will use to handle hypermetric inequalities on at least 4 points cannot be applied to the triangle inequalities.

Consider the set of vectors $\{\mathbf{z}_i\}$, $i \ge 1$. For each coordinate, the vectors in this subset take on at most 2 different values, and hence this subset is ℓ_1-embeddable. Therefore, any ℓ_1 inequality (and in particular any hypermetric inequality) not involving $\mathbf{z}_0$ must be satisfied.

Now let $B = \mathbf{b}\mathbf{b}^t \in \mathcal{C}^*$, where $\mathbf{b} \in \mathbb{Z}^{k+1}$ and $\sum_{i=0}^k b_i = 1$, be any hypermetric inequality supported on $r = k+1$ points. By the above discussion, it suffices to consider the case where $\mathbf{z}_0$ is one of the points, and we can assume that the points are $0, 1, \ldots, k$. Our goal is to show that $\sum_{i<j\le k} B_{ij} \|\mathbf{z}_i - \mathbf{z}_j\|^2 \le 0$. By definition, for $i, j \ge 1$,

$$\|\mathbf{z}_i - \mathbf{z}_j\|^2 = 2 - 2(4\epsilon^2 + (1-4\epsilon^2)P(u_i \cdot u_j)) = 2(1-4\epsilon^2)(1 - P(u_i \cdot u_j)) \ ,$$

and $\|\mathbf{z}_i - \mathbf{z}_0\|^2 = 2 - 4\epsilon$. Hence,

$$\sum_{0\le i<j\le k} B_{ij}\|\mathbf{z}_i - \mathbf{z}_j\|^2 = 2(1-2\epsilon)\sum_{i=1}^k B_{0i} + 2(1-4\epsilon^2)\sum_{0<i<j\le k} B_{ij}(1 - P(u_i \cdot u_j)) \ .$$

Therefore, we need to show

$$\sum_{i=1}^k B_{0i} + (1+2\epsilon)\sum_{0<i<j\le k} B_{ij}(1 - P(u_i \cdot u_j)) \le 0 \ . \tag{3}$$

We will require the technical lemma below, but first some definitions. By homogeneity we may assume $b_0 < 0$ (and hence that $b_0 \le -1$ since $b_0 \in \mathbb{Z}$). Let

$$S = \{i \in [k] : b_i > 0\} \ ,$$
$$T = \{i \in [k] : b_i < 0\} \ .$$

Next let $H_{ij} = (\mathbf{u}_i \cdot \mathbf{u}_j + 1)(\mathbf{u}_i \cdot \mathbf{u}_j)^{\frac{2m}{\gamma}}$ and $M_{ij} = (\mathbf{u}_i \cdot \mathbf{u}_j)^{\frac{1}{\gamma}}$, and let Δ_{ij} be the Hamming distance between $\mathbf{u}_i$ and $\mathbf{u}_j$. With these definitions we can then write $P(\mathbf{u}_i \cdot \mathbf{u}_j) = \beta H_{ij} + \alpha M_{ij} + (1 - \alpha - 2\beta)(1 - \frac{2}{m}\Delta_{ij})$.

Lemma 2. *Assume that γ, ϵ and m satisfy the conditions in Lemma 1. Then,*

1. $\sum_{0<i<j\le k} B_{ij} = \frac{1}{2}((1-b_0)^2 - \sum_{i=1}^k b_i^2)$
2. $\sum_{0<i<j\le k} B_{ij}(-\beta H_{ij} - \alpha M_{ij}) \le 15\gamma \sum_{i\in S, j\in T} b_i(-b_j)$
3. $\sum_{0<i<j\le k} B_{ij}\Delta_{ij} \le \frac{1}{4}m(1-b_0)^2$

Proof. The first equality is an immediate consequence of the fact that $\sum_{i=1}^k b_i = 1 - b_0$ and that $(\sum_{i=1}^k b_i)^2 = \sum_{i=1}^k b_i^2 + 2\sum_{0<i<j\le k} b_i b_j$.

For the second inequality, note first that $\mathbf{u}_i \cdot \mathbf{u}_j \le 1 - 1/m$. Hence, H_{ij} is negligible for all $i \ne j$. Moreover, since the $\mathbf{u}_i$ are unit vectors and $1/\gamma$ is even, it follows that $0 \le M_{ij} \le 1$. Hence, by the bounds for α and β given by Lemma 1 it follows that $\beta H_{ij} + \alpha M_{ij} \le 15\gamma$ and the second inequality follows.

For the last inequality notice that since $\varDelta_{ij}$ is the sum of m cut metrics (defined by the m coordinates), it is enough to show that for every subset $I \subset \{0, 1, \ldots, k\}$,

$$\sum_{0<i<j\leq k} B_{ij}\delta_I(i,j) \leq \frac{1}{4}(1-b_0)^2 \ .$$

Indeed, using the fact that B is a hypermetric we have,

$$\sum_{0<i<j\leq k} B_{ij}\delta_I(i,j) = \sum_{i\in I, j\notin I} b_i b_j = \left(\sum_{i\in I} b_i\right)\cdot\left(1 - b_0 - \sum_{i\in I} b_i\right) \leq \left(\frac{1-b_0}{2}\right)^2 \ .$$

□

We can now bound the left-hand-side of (3). To begin with, we have,

$$\begin{aligned}
&\sum_{i=1}^{k} B_{0i} + (1+2\epsilon)\sum_{0<i<j\leq k} B_{ij}(1 - P(u_i\cdot u_j)) \\
=&\sum_{i=1}^{k} B_{0i} + (1+2\epsilon)\sum_{0<i<j\leq k} B_{ij}(1 - \beta H_{ij} - \alpha M_{ij} - (1-\alpha-2\beta)(1-\frac{2}{m}\varDelta_{ij})) \\
=&\sum_{i=1}^{k} B_{0i} + (1+2\epsilon)\sum_{0<i<j\leq k} B_{ij}(-\beta H_{ij} - \alpha M_{ij} + \alpha + 2\beta + (1-\alpha-2\beta)\frac{2}{m}\varDelta_{ij}) \ .
\end{aligned}$$

Applying the inequalities from Lemma 2 it then follows that the above is upper-bounded by

$$\begin{aligned}
&b_0(1-b_0) + (1+2\epsilon)\left(15\gamma\sum_{i\in S, j\in T} b_i(-b_j) + \frac{1}{2}(\alpha+2\beta)\left[(1-b_0)^2 - \sum_{i=1}^{k} b_i^2\right] + \frac{1}{2}(1-\alpha-2\beta)(1-b_0)^2\right) \\
&= \frac{1}{2}(1-b_0^2) + 2\epsilon\frac{1}{2}(1-b_0)^2 + (1+2\epsilon)\left(15\gamma\sum_{i\in S, j\in T} b_i(-b_j) - \frac{1}{2}(\alpha+2\beta)\sum_{i=1}^{k} b_i^2\right) \\
&< \frac{1}{2}(1-b_0^2) + 2\epsilon\frac{1}{2}(1-b_0)^2 + (1+2\epsilon)\left(15\gamma\sum_{i\in S, j\in T} b_i(-b_j) - \left[\frac{2\epsilon}{1+2\epsilon} - 2\gamma\right]\sum_{i=1}^{k} b_i^2\right) \\
&= \frac{1}{2}(1-b_0^2) + 2\epsilon\frac{1}{2}(1-b_0)^2 - 2\epsilon\sum_{i=1}^{k} b_i^2 + (1+2\epsilon)\left(15\gamma\sum_{i\in S, j\in T} b_i(-b_j) + 2\gamma\sum_{i=1}^{k} b_i^2\right) \\
&< \frac{1}{2}(1-b_0^2) - \epsilon(2\sum_{i=1}^{k} b_i^2 - (1-b_0)^2) + 15\gamma(1+2\epsilon)\left(\sum_{i\in S, j\in T} b_i(-b_j) + \sum_{i=1}^{k} b_i^2\right) \\
&< \frac{1}{2}(1-b_0^2) - \epsilon(2\sum_{i=1}^{k} b_i^2 - (1-b_0)^2) + 30\gamma\left(\sum_{i\in S, j\in T} b_i(-b_j) + \sum_{i=1}^{k} b_i^2\right) \ .
\end{aligned}$$

Note that since the hypermetric inequality we are considering is *not* a triangle inequality, it follows that we must have $\sum_{i>0} b_i^2 \geq 3$. But then, the following technical lemma can be used to show that the above is bounded by 0, completing the proof of the theorem.

Lemma 3. *Let $k \leq \frac{2}{45}\frac{\epsilon}{\gamma} - 1$ and let $0 < \epsilon < \frac{1}{6}$ and $\gamma > 0$. Assume $b_0 \leq -1$, $\sum b_i^2 \geq 3$ and that $b_i \neq 0$ for all i. Then*

$$\frac{1}{2}(1-b_0^2) - 2\epsilon\left(\sum_{i=1}^{k} b_i^2 - \frac{1}{2}(1-b_0)^2\right) + 30\gamma\left(\sum_{i\in S, j\in T} b_i(-b_j) + \sum_{i=1}^{k} b_i^2\right) < 0 \ .$$

Proof. It is not hard to to check that since $\epsilon < \frac{1}{6}$ and b_0 is a (strictly) negative integer, we have

$$\frac{1}{2}(1-b_0^2) - 2\epsilon\left(\sum_{i=1}^{k} b_i^2 - \frac{1}{2}(1-b_0)^2\right) \leq -2\epsilon\left(\sum_{i=1}^{k} b_i^2 - 2\right) \leq -\frac{2\epsilon}{3}\sum_{i=1}^{k} b_i^2 \ .$$

Note that it was critical to have $\sum_{i>0} b_i^2 \geq 3$ here, as only then can we claim that $\sum_{i=1}^{k} b_i^2 - 2$ is a positive constant. Indeed, for the triangle inequality ($b_0 = -1, b_1 = b_2 = 1$), i.e., the only hypermetric inequality for which this doesn't hold, we cannot expect any method bounding the slack of the inequality to work: the VERTEX COVER edge constraints force the triangle inequality to be tight for edges!

It now suffices to prove that

$$-\frac{2\epsilon}{3}\sum_{i=1}^{k} b_i^2 + 30\gamma\left(\sum_{i\in S, j\in T} b_i(-b_j) + \sum_{i=1}^{k} b_i^2\right) < 0 \ . \tag{4}$$

Let s, t be the cardinalities of S, T, respectively, and let $x = \sum_{i\in S} b_i$ and $y = \sum_{i\in T}(-b_i)$. Now, using the Cauchy-Schwartz inequality and the fact that $s, t \leq k$, we get

$$\frac{\sum_{i\in S, j\in T} b_i(-b_j) + \sum_{i=1}^{k} b_i^2}{\sum_{i=1}^{k} b_i^2} \leq 1 + \frac{xy}{s(x/s)^2 + t(y/t)^2} \leq 1 + k\frac{xy}{x^2+y^2} \leq 1 + k/2 \ .$$

(Note that if $y = t = 0$ the bound is trivial and we therefore ignored this case above.) Hence,

$$-\frac{2\epsilon}{3}\sum_{i=1}^{k} b_i^2 + 30\gamma\left(\sum_{i\in S, j\in T} b_i(-b_j) + \sum_{i=1}^{k} b_i^2\right) < \left(-\frac{2\epsilon}{3} + 30\gamma(1+k/2)\right)\sum_{i=1}^{k} b_i^2 \ ,$$

and so (4) holds as long as $k \leq \frac{2}{45}\frac{\epsilon}{\gamma} - 1$. □

Theorem 2 now follows. □

4 Hypermetric Inequalities vs. Lovász-Schrijver SDP Lift-and-Project

In this section we show that hypermetric inequalities need not be derived by Lovász and Schrijver's LS_+ lift-and-project system. Our plan of attack is as follows. After stating all necessary definitions, we will first show that no pure hypermetric inequalities are derived by LS_+ for the convex cone defined by the inequalities $0 \leq x_i \leq x_0$, $i = 1, \ldots, n$. We will then use this result to show the following for VERTEX COVER: Fix a graph G and an independent set S in G, and consider a VERTEX COVER SDP for G derived using LS_+ lift-and-project. Then the constraints defining this SDP do not imply *any* of the pure hypermetric constraints supported on S.

We begin by defining the Lovász-Schrijver LS_+ lift-and-project system [18]. In what follows all vectors will be indexed starting at 0. Recall that a set $C \subset \mathbb{R}^n$ is a convex cone if for every $\mathbf{y}, \mathbf{z} \in C$ and for every $\alpha, \beta \geq 0$, $\alpha\mathbf{y} + \beta\mathbf{z} \in C$. Given a convex cone $C \subset \mathbb{R}^{n+1}$ we denote its projection onto the hyperplane $x_0 = 1$ by $C|_{x_0=1}$. Let $\mathbf{e}_i$ denote the vector with 1 in coordinate i and 0 everywhere else. Let $Q_n \subset \mathbb{R}^{n+1}$ be the convex cone defined by the constraints $0 \leq x_i \leq x_0$ and fix a convex cone $C \subset Q_n$. The lifted cone $M_+(C) \subseteq \mathbb{R}^{(n+1)\times(n+1)}$ consists of all positive semidefinite matrices $(n+1)\times(n+1)$ matrices Y such that, Property I. For all $i = 0, 1, \ldots, n$, $Y_{0i} = Y_{ii}$.

Property II. For all $i = 0, 1, \ldots, n$, $Y\mathbf{e}_i, Y\mathbf{e}_0 - Y\mathbf{e}_i \in C$.

The cone $M_+(C)$ is the LS_+ *positive semidefinite tightening* for C. This procedure can be iterated by projecting $M_+(C)$ back to $\mathbb{R}^{n+1}$ and then re-applying the M_+ operator to the projection. In particular, let $N_+(C) = \{Y\mathbf{e}_0 : Y \in M_+(C)\} \subseteq \mathbb{R}^{n+1}$. Define $N_+^k(C)$ inductively by setting $N_+^0(C) = C$ and $N_+^k(C) = N_+(N_+^{k-1}(C))$, and define $M_+^k(C)$ to be $M_+(N_+^{k-1}(C))$. Lovász and Schrijver show that $N_+^{k+1}(C) \subseteq N_+^k(C)$ and $M_+^{k+1}(C) \subseteq M_+^k(C)$ and that moreover these containment are proper if and only if $N_+^k(C)|_{x_0=1}$ is not the integral hull of $C|_{x_0=1}$. Moreover, they show that $N_+^n(C)|_{x_0=1}$ is equal to the integral hull of $C|_{x_0=1}$. It is useful to note, that $Y \in M_+^k(C) \subseteq \mathbb{R}^{(n+1)\times(n+1)}$ if and only if Y is PSD and satisfies both Property I and the following property:

Property II$'$. For all $i = 0, 1, \ldots, n$, $Y\mathbf{e}_i, Y\mathbf{e}_0 - Y\mathbf{e}_i \in N_+^{k-1}(C)$.

With these definitions in hand, we can now begin by showing that $M_+(Q_n)$ does not satisfy any pure hypermetric constraint (recall that Q_n is the cone satisfying $0 \le x_i \le x_0$ for all $i = 1, \ldots, n$). As a warm up we examine the triangle inequality of SDP (1) for a three vertex graph with no edges. Note that this SDP has no edge constraints. Moreover, any vector solution $\mathbf{v}_i$ can be mapped using the affine transformation $\mathbf{v}_i \to (\mathbf{v}_i + \mathbf{v}_0)/2$ to a set of vectors whose Gram matrix is in $M_+(Q_3)$, and vice versa. Now consider three vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ that correspond to the three vertices of the graph. Geometrically it is possible to place these vectors such that the Gram matrix of $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ satisfies Properties I and II above for an LS_+ tightening, yet $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ violate triangle inequality. We can accomplish this by making $\mathbf{v}_1$ and $\mathbf{v}_2$ almost coincide and placing $\mathbf{v}_3$ between them.

Our counterexample for hypermetrics will be a generalization of the following more subtle matrix in $M_+(Q_3)$ violating triangle inequality:

$$Y = \begin{pmatrix} 1 & \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & 0 & \beta\epsilon \\ \epsilon & 0 & \epsilon & \beta\epsilon \\ \epsilon & \beta\epsilon & \beta\epsilon & \epsilon \end{pmatrix} .$$

By having $\epsilon \in (0, 1/2)$ and $\beta \in [0, 1]$ we ensure Y satisfies Properties I and II. One can show that by setting ϵ arbitrarily close to 0 and β close to but bigger than $1/2$, we ensure that Y is PSD, while ensuring that its Cholesky decomposition violates the triangle inequality. This matrix sacrifices some of the above geometric intuition to make our calculations easier.

This construction can be extended to show that $M_+(Q_n)$ does not satisfy any inequality $\sum b_i b_j d_{ij} \le 0$ where b is a vector of length $n = 2k+1$, $\sum b_i = 1$, and for all i, $|b_i| = 1$. Indeed, consider an inequality on $2k+1$ points defined by the vector $(0, b_1, b_2, \ldots, b_{2k+1}) \in \mathbb{Z}_+^{2k+2}$ (note that $b_0 = 0$) where $b_i = 1$ for $i = 1, \ldots, k+1$ and $b_i = -1$ for $i = k+2, \ldots, 2k+2$. In this way we naturally split the points into two clusters of size $k+1$ and k points. The associated inequality requires that the sum of distances across the clusters dominates the sum of distances within the clusters. Define the distance within the clusters as 2ϵ, and the distance across the clusters as $2\epsilon(1-\beta)$. We have $k(k+1)$ cross pairs and $\binom{k}{2} + \binom{k+1}{2} = k^2$ inner pairs. Therefore in order to violate the inequality, we should have $2\epsilon(1-\beta)k(k+1) < 2\epsilon k^2$. In other words it suffices for β to be slightly bigger than $\frac{1}{k+1}$ (this will be crucial later).

Define the matrix

$$Y^{(s,t)} = \begin{pmatrix} 1 & \epsilon J_{1,s} & \epsilon J_{1,t} \\ \epsilon J_{s,1} & \epsilon I_s & \epsilon\beta J_{s,t} \\ \epsilon J_{t,1} & \epsilon\beta J_{t,s} & \epsilon I_t \end{pmatrix} ,$$

where $J_{m,n}$ is the $m \times n$ all-1 matrix, and I_n is the $n \times n$ identity matrix (note that $s = 2, t = 1$ gives Y above). The configuration described above can be realized by the matrix $Y^{(k+1,k)}$ of order $(2k+2)$. Similarly as in the case of the triangle inequality, $Y^{(s,t)}$ satisfies Properties I and II as long as $\epsilon \in (0, 1/2)$ and $b \in [0, 1]$.

Hence, $Y^{(k+1,k)}$ is in $M_+(Q_n)$ provided we can show that it is PSD. This is implied by the following technical lemma.

Lemma 4. *For all s, t, such that $s + t = 2k + 1$ there exist $\epsilon \in (0, 1/2)$ and $\beta > \frac{1}{k+1}$ such that the matrix $Y^{(s,t)} \in \mathbb{R}^{(2k+2)\times(2k+2)}$ is PSD.*

Proof. To simplify notation, we will denote $\frac{1}{\epsilon}Y^{(s,t,\epsilon,\beta)}$ by $Y^{(s,t)}$.

We begin by computing all principal minors of $Y^{(s,t)}$. Subtracting the third row from the second in $Y^{(s,t)}$, we get $\det(Y^{(s,t)}) = \det(Y^{(s-1,t)}) + \det(L^{(s-1,t)})$, where $L^{(s,t)}$ is the same matrix as $Y^{(s,t)}$ except that $L^{(s,t)}_{22} = 0$ (instead of 1).

The same operation on rows shows that $\det(L^{(s,t)}) = \det(L^{(1,t)})$. Next denote by $K^{(1,t)}$ the same matrix as $L^{(1,t)}$ except that $K^{(1,t)}_{33} = 0$ (instead of 1). Subtracting the fourth row from the third in $L^{(1,t)}$ we get $\det(L^{(1,t)}) = \det(L^{(1,t-1)}) + \det(K^{(1,t-1)})$ where again $\det(K^{(1,t)}) = \det(K^{(1,1)})$. Finally let $M^{(1,t)}$ be the same matrix as $Y^{(1,t)}$ except that $M^{(1,t)}_{33} = 0$ (instead of 1). Again, the same row operation in $Y^{(1,t)}$ gives $\det(Y^{(1,t)}) = \det(Y^{(1,t-1)}) + \det(M^{(1,t-1)})$ with $\det(M^{(1,t)}) = \det(M^{(1,1)})$.

For simplicity denote $Y^{(1,1)}, L^{(1,1)}, M^{(1,1)}, K^{(1,1)}$ by Y, L, M, K respectively. Then for these base matrices

$$Y = \begin{pmatrix} 1/\epsilon & 1 & 1 \\ 1 & 1 & \beta \\ 1 & \beta & 1 \end{pmatrix} \; L = \begin{pmatrix} 1/\epsilon & 1 & 1 \\ 1 & 0 & \beta \\ 1 & \beta & 1 \end{pmatrix} \; M = \begin{pmatrix} 1/\epsilon & 1 & 1 \\ 1 & 1 & \beta \\ 1 & \beta & 0 \end{pmatrix} \; K = \begin{pmatrix} 1/\epsilon & 1 & 1 \\ 1 & 0 & \beta \\ 1 & \beta & 0 \end{pmatrix}$$

we have

$$\begin{aligned} \det(Y) &= \frac{1}{\epsilon}(1 - \beta^2 + 2\beta\epsilon - 2\epsilon) \\ \det(L) = \det(M) &= \frac{1}{\epsilon}(-\beta^2 + 2\beta\epsilon - \epsilon) \\ \det(K) &= \frac{\beta}{\epsilon}(-\beta^2 + 2\epsilon) \ . \end{aligned}$$

Using these values, we have

$$\begin{aligned} \det(Y^{(s,t)}) &= \det(Y^{(1,t)}) + (s-1)\left(\det(L) + (t-1)\det(K)\right) \\ &= \det(Y) + (t-1)\det(M) + (s-1)\left(\det(L) + (t-1)\det(K)\right) \\ &= \frac{1}{\epsilon}(1 - st\beta^2 + \epsilon(-t - s + 2st\beta)) \ . \end{aligned}$$

Recall that we required $\beta > \frac{1}{k+1}$ and so we can take β arbitrarily close to that bound. But then

$$st\beta^2 \leq \left(\frac{2k+1}{2}\right)^2 \frac{1}{(k+1)^2} = \left(\frac{2k+1}{2k+2}\right)^2 < 1$$

making $\det(Y^{(s,t)})$ strictly positive for sufficient small ϵ.

□

We are ready now to show that VERTEX COVER SDPs in the LS_+ hierarchy violate pure hypermetrics on any independent set. Fix an n-vertex graph $G = (V, E)$ and consider the convex cone $C \subset Q_n$ consisting of all vectors $x \in \mathbb{R}^{n+1}$ such that $x_i + x_j \geq x_0$. Then LS_+ lifting yields the following sequence of SDPs for G: $M_+(C), M_+^2(C), \ldots$. We will show that for all k, every independent set S in G, and all pure hypermetrics B supported on S, there exists $Y \in M_+^k(C)$ such that Y does not satisfy B.

To that end, fix k and S, and let $s = |S|$ be odd. Without loss of generality, assume that $S = \{1, 2, \ldots, s\}$. Fix a pure hypermetric B defined on the set S. By the discussion above we know that there exists $Y' \in M_+(Q_s)$ that violates the pure hypermetric B. Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_s$ be the Cholesky decomposition for Y'. Now let $Y \in \mathbb{R}^{(n+1)\times(n+1)}$ be the matrix with Cholesky decomposition $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_s, \mathbf{v}'_{s+1}, \ldots, \mathbf{v}'_n$ where $\mathbf{v}'_j = \mathbf{v}_0$ for all $j \geq s+1$. By construction Y is PSD, satisfies Property I, and does not satisfy B on S. So it suffices to verify Property II′ in order to show that $Y \in M_+^k(C)$. Note that $Y\mathbf{e}_i$ is the all-1 vector for all $i \geq s+1$

and hence Property II$'$ holds for all $i \geq s+1$ since the all-1 vector is in the integral hull and hence in $N_+^k(C)$ for all k. Now consider a vector $Y\mathbf{e}_i$ where $1 \leq i \leq s$. Note that $Y_{00} = Y_{0j}$ for all $j \geq s+1$. But then, since S is independent, it follows that the projection of $Y\mathbf{e}_i$ onto the hyperplane $x_0 = 1$ is also in the integral hull and hence in $N_+^k(C)$. Similarly, it follows that $Y(\mathbf{e}_0 - \mathbf{e}_i)$ is also in $N_+^k(C)$ whenever $1 \leq i \leq s$. So Property II' holds for all i, and $Y \in M_+^k(C)$.

We end this section by remarking that the above arguments can be combined with those from [10] to show that there is a graph G for which $O(\sqrt{\log n/\log\log n})$ rounds of LS_+ produce an SDP which (a) does not satisfy the triangle inequality *and* (b) has integrality gap $2 - o(1)$. The argument, which we do not have room to go into here, considers the Frankl-Rödl graph G_m^γ to which we append three isolated vertices. The idea is to not satisfy the triangle inequality on the isolated vertices while the remaining vertices will essentially employ the SDP solutions from [10].

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