Question 1. [3 MARKS]

Suppose a, b, c and d are boolean variables.
Give a logical expression (not using `if`) that represents the value returned by the following Java code:

```
if (a) {
  return b;
} else if (c) {
  return d;
} else {
  return false;
}
```

Solution: $(a \wedge b) \vee (\sim a \wedge c \wedge d)$

Question 2. [5 MARKS]

Part (a) [2 MARKS]

Consider the following property of natural numbers:
$m(x, y, z) =$ "$x * y = z$".
Using the property m, write a sentence that is true if and only if a exactly divides b.

Solution: $\exists y \in N, m(a, y, b)$

Grading: 1 for correct quantifier
1 for correct parameters for m

Part (b) [3 MARKS]

Let $P(x)$ denote the property "natural number x is prime."
Write a sentence that says there is a minimum prime number.
Use P, but don't use max or min.

Solution: $\exists y \in N, [P(y) \wedge (\forall x \in N, P(x) \implies x > y)]$.

Grading: 1 mark off for each mistake

Question 3. [6 MARKS]

Three teens went shopping one afternoon and each bought 2 items.
Al bought a CD and a muffin.
Flo bought a book and a CD.
Lee bought a T-shirt and a postcard.

Let $\mathcal{T} = \{\text{Al, Flo, Lee}\}$ and let $\mathcal{I} = \{\text{book, CD, muffin, postcard, T-shirt}\}$.
For $t \in \mathcal{T}$ and $x \in \mathcal{I}$, let $bought(t, x)$ denote "t bought item x".

Part (a) [1 MARK]

Write the following statement in everyday language: $\exists x \in \mathcal{I}, \forall t \in \mathcal{T}, bought(t, x)$.

Solution: There is some item that all the teens bought.

Part (b) [1 MARK]

Is the sentence in (a) true or false?

Solution: false

Part (c) [4 MARKS]

Write a single open sentence that is
true for $t =$ Lee, and
false for $t =$ Al and $t =$ Flo, and
does not use any constants (i.e. Al, Flo, Lee, book, CD, muffin, postcard or T-shirt).

Solution: $\forall x \in \mathcal{I}, [bought(t, x) \implies \forall s \in \mathcal{T}, (s \neq t \implies \sim bought(s, x))]$

Grading: 1 mark off for each mistake

Question 4. [12 MARKS]

Let N be the set of positive integers. Consider the sentence:

$$\exists i \in N, \forall j \in N, a_{j+i} = a_j$$

Part (a) [1 MARK]

State whether the sentence is true or false for the following sequence:

$$1, 2, 3, 1, 2, 3, 1, 2, 3, \ldots.$$

Part (b) [11 MARKS]

Formally prove, using the proof structures discussed in class, that the sentence is false for the following sequence:

$$1, 1, 2, 1, 2, 1, 2, 1, 2, \ldots$$

Use the following formula for this sequence:

$$a_n = \begin{cases} 1, & n = 1 \lor n \text{ is even} \\ 2, & n \neq 1 \land n \text{ is odd} \end{cases} .$$

(More room for your solution to question 4(b))

Question 5. [14 MARKS]

Let $\mathcal{F}$ denote the set of all functions with domain N and codomain $R^{\geq 0}$.

For any function $f \in \mathcal{F}$, let

$$\Gamma(f) = \{g \in \mathcal{F} \mid \exists a \in R, \exists d \in R, \forall n \in N, g(n) \leq a \cdot f(n) + d\}.$$

Formally prove, using the proof structures discussed in class, that $\forall f \in \mathcal{F}, O(f) \subseteq \Gamma(f)$.

You may use the following result (recall that R^+ represents the set of positive real numbers):

$$O(f) = \{g \in \mathcal{F} \mid \exists c \in R^+, \exists b \in N, \forall n \in N, (n \geq b \Rightarrow g(n) \leq c \cdot f(n))\}.$$

Solution:

Let $f \in \mathcal{F}$ be arbitrary.
  Let $g \in O(f)$.
    Then $\exists c \in R, \exists b \in N, \forall n \in N, (n \geq b \implies g(n) \leq c \cdot f(n))$.
    Let $c \in R$ and $b \in N$ be such that $\forall n \in N, (n \geq b \implies g(n) \leq c \cdot f(n))$.
    Let $a = c$ and let $d = \max(\{g(n) - c \cdot f(n) \mid n < b\} \cup \{0\})$.
    Then $a \in R$ and $d \in R$.
    Let $n \in N$ be arbitrary.
      Case 1: $n < b$
          Then, by definition of d, $g(n) - a \cdot f(n) \leq d$.
          Thus $g(n) \leq a \cdot f(n) + d$.
      Case 2: $n \geq b$
          Since $n \geq b \implies g(n) \leq c \cdot f(n)$,
          it follows that $g(n) \leq c \cdot f(n) = a \cdot f(n)$.
          By definition, $d \geq 0$.
          Therefore $g(n) \leq a \cdot f(n) + d$.
      Hence $g(n) \leq a \cdot f(n) + d$.
      Since n is an arbitrary element of N,
    $\forall n \in N, g(n) \leq a \cdot f(n) + d$.
    Thus $\exists a \in R, \exists d \in R, \forall n \in N, g(n) \leq a \cdot f(n) + d$.
  Therefore $g \in \Gamma(f)$.
  Thus $O(f) \subseteq \Gamma(f)$.
  Since f is an arbitrary element of $\mathcal{F}$,
$\forall f \in \mathcal{F}, O(f) \subseteq \Gamma(f)$.

Grading:

(More room for you solution to question 5).

Question 6. [6 MARKS]

Consider the following algorithm that evaluates a degree n polynomial at a point x.

```
EVAL(A,n,x)
v = A[0]
y = 1
i = 1
while i<=n do
     y = y*x
     if A[i] != 0 then
       v = v + y * A[i]
     i = i+1
return v
```

Part (a) [2 MARKS]

State a loop invariant for **EVAL** involving A, v and i. Do NOT prove that it is a loop invariant.

Solution: $v = A[0]x^0 + A[1]x^1 + \cdots + A[i-1]x^{i-1}$.

Grading: 1 mark for getting a polynomial
1 mark for getting the last term correct.

Part (b) [4 MARKS]

Let t_E denote the worst case time complexity of EVAL(A,n,x) as a function of n, where 1 step is an addition, subtraction, or multiplication. Determine, with justification, t_E. Do NOT use O, Ω or Θ.

Solution: There are n complete iterations. In each, at most 4 arithmetic operations are performed. Therefore, $t_E(n) \leq 4n$.

If all entries in A are nonzero, then exactly 4 iterations are performed, so $t_E(n) \geq 4n$. Thus $t_E(n) = 4n$.

Grading: 1 mark for correct answer
1 mark for doing both upper and lower
1 mark for upper bound justification
1 mark for lower bound justification.

Question 7. [4 MARKS]

Consider the floating-point number system with $\beta = 2, p = 3, e_{min} = -3$ and $e_{max} = 3$.
Express the base 10 number 0.6 in this system, using rounding to nearest. Explain your reasoning.
You might find the following useful: $1/8 = 0.125, 1/16 = 0.0625$

Solution: 0.6 is between $0.5 + 0 + 0$ and $0.5 + 0 + 0.125$.
The nearest is $0.5 + 0 + 0.125 = 1.01x2^{-1}$.

Question 8. [5 MARKS]

Let $f : R \to R$ be defined by $f(x) = 1 - x$.

Part (a) [2 MARKS]

What is the condition number of f?

Solution: $|(f(x') - f(x))/f(x)/[(x' - x)/x]| = |x/(1 - x)|$

Part (b) [3 MARKS]

How is your answer to part (a) related to catastrophic cancellation?

Solution: For x near 1, the condition number is large so error in x will be amplified by f. This is also explained by catastrophic cancellation: for x near 1, $f(x)$ is the difference between two close numbers.

Question 9. [7 MARKS]

Formally prove, using the proof structures discussed in class, that for all positive real numbers c and all real numbers x, if $0 < rel(1/c, x) < 1$ then $rel(1/c, 2x - cx^2) < rel(1/c, x)$.
(Recall that $rel(x, x')$ is the relative error when approximating x by x').

Solution: Let c be an arbitrary positive real number and let x be an arbitrary real number. The relative error when using x as an approximation for $1/c$ is $|(x - 1/c)/(1/c)| = |cx - 1|$.

Suppose that $0 < |cx - 1| < 1$.

Then the relative error when using $2x - cx^2$ as as an approximation for $1/c$ is $|(2x - cx^2 - 1/c)/(1/c)| = |2cx - (cx)^2 - 1| = |(cx)^2 - 2cx + 1| = |(cx - 1)^2| = |cx - 1| \cdot |cx - 1| < |cx - 1|$, since $0 < |cx - 1| < 1$.

Hence, for all positive real numbers c, if the relative error when using x as an approximation for $1/c$ is larger than 0, but smaller than 1, then the relative error when using $2x - cx^2$ as an approximation for $1/c$ is strictly smaller.

Total Marks = 62