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UNIVERSITY OF TORONTO
Faculty of Arts and Science

DECEMBER 2003 EXAMINATIONS

CSC 165H1 F
Instructor: François Pitt

Duration — 3 hours

Examination Aids: **One 8.5" × 11" sheet of paper, *hand*written on both sides; one *non-programmable* calculator.**

Student Number: |_|_|_|_|_|_|_|_|_|

Last (Family) Name(s): ______________________________

First (Given) Name(s): ______________________________

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*Do **not** turn this page until you have received the signal to start.*
(In the meantime, please fill out the identification section above, and read the instructions below *carefully*.)

This final examination consists of 7 questions on 18 pages (including this one), printed on both sides of the paper. *When you receive the signal to start, please make sure that your copy of the examination is complete.*

Answer each question directly on the examination paper, in the space provided. Some of the pages are "blank" (*i.e.*, they do not contain any question): use them for rough work. If you need more space for one of your solutions, use one of the blank pages and *indicate clearly the part of your work that should be marked.*

If you are unable to answer a question (or part of a question), you will get 20% of the marks for the question (or part of the question) if you state clearly that you do not know how to answer. Note that you will *not* get those marks if your answer contains contradictory statements (such as "I do not know how to answer" followed or preceded by parts of a solution that have not been crossed off).

General Hint: We were careful to leave ample space on the examination paper to answer each question.

Marking Guide

# 1:	______/ 30
# 2:	______/ 20
# 3:	______/ 25
# 4:	______/ 10
# 5:	______/ 20
# 6:	______/ 10
# 7:	______/ 35
Bonus Marks:	______/ 3
TOTAL:	______/150

Good Luck!

*[This is a "blank" page. Use the space below for rough work. This page will **not** be marked, unless you clearly indicate the part of your work that you want us to mark.]*

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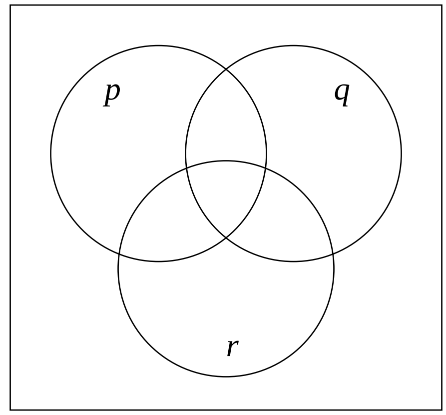
Question 1. [30 MARKS]

Part (a) [6 MARKS]

Consider the following sentence:

($S1$) r is sufficient for p to be equivalent to q

In the space below, rewrite ($S1$) using symbolic notation, and in the diagram on the right, write an "X" in every region where sentence ($S1$) is false.

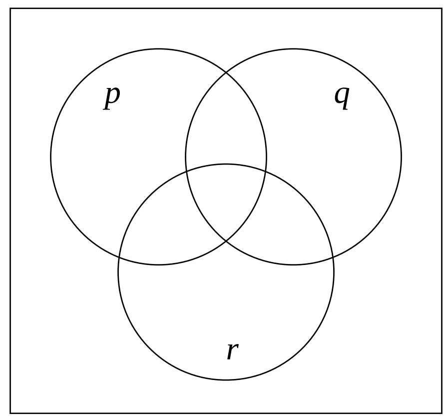


Part (b) [6 MARKS]

Consider the following sentence:

($S2$) it is necessary that p imply q in order for p and r to be true

In the space below, rewrite ($S2$) using symbolic notation, and in the diagram on the right, write an "X" in every region where sentence ($S2$) is false.



Part (c) [5 MARKS]

A function $f : \mathbb{R} \to \mathbb{R}$ is *continuous* at point $a \in \mathbb{R}$ iff for every positive $\varepsilon \in \mathbb{R}$, there is some positive $\delta \in \mathbb{R}$ such that for all $x \in \mathbb{R}$, if $|x - a| < \delta$, then $|f(x) - f(a)| < \varepsilon$.

Write a natural English sentence to express the fact that f is *not* continuous at point a.

Question 1. (continued)

Part (d) [5 marks]

Let T denote the set of all instants in time (*e.g.*, "2:43:15pm on Tuesday 2 December 2003"), and P denote the set of all people. Let $f(x, y)$ represent the sentence: "person x is a friend of person y" and $h(x, t)$ represent the sentence: "person x is happy at time t".

Using only the domains and predicates above, write a symbolic formula to represent the *negation* of the sentence: "Everybody has a friend who is always happy." (Do *not* simply put a negation in front of the sentence!)

Part (e) [4 marks]

Let $\mathbb{R}$ denote the set of real numbers. Using standard notation like $x = y$, $x \neq y$, $x < y$, etc., write a symbolic formula to represent the sentence: "No positive number is equal to any negative number."

Part (f) [4 marks]

For an arbitrary predicate $p(x)$ and domain D, we express the sentence: "there is exactly one $x \in D$ such that $p(x)$" with the symbolic formula: $\exists x \in D, \big(p(x) \wedge \forall y \in D, p(y) \rightarrow y = x\big)$.

Write a symbolic formula to express the sentence: "There are exactly *two* $x \in D$ such that $p(x)$."

Question 2. [20 MARKS]

Consider the following statement about sequences of natural numbers $a_0, a_1, a_2, \ldots$:

$$(S) \quad \forall i \in \mathbb{N}, i > 0 \rightarrow \big((\exists j \in \mathbb{N}, j < i \wedge a_j < a_i) \vee (\exists j \in \mathbb{N}, j > i \wedge a_j > a_i)\big)$$

Part (a) [8 MARKS]

Define a sequence for which (S) is true and write a carefully structured proof that (S) is true for your sequence.

Question 2. (CONTINUED)

Part (b) [12 MARKS]

Define a sequence for which (S) is false and write a carefully structured proof that (S) is false for your sequence.

Question 3. [25 MARKS]

Consider the following algorithm that computes the matrix product of the two $n \times n$ arrays A and B.

```
PROD(A, B, n):
1.   i := 0                                       // _______ steps
2.   while i < n:                                 // _______ steps
3.       j := 0                                   // _______ steps
4.       while j < n:                             // _______ steps
5.           C[i][j] := 0                         // _______ steps
6.           k := 0                               // _______ steps
7.           while k < n:                         // _______ steps
8.               C[i][j] := C[i][j] + (A[i][k] × B[k][j])   // _______ steps
9.               k := k + 1                       // _______ steps
10.          j := j + 1                           // _______ steps
11.      i := i + 1                               // _______ steps
12.  return C                                     // _______ steps
```

Part (a) [6 MARKS]

Write the exact number of steps performed during each line of the algorithm, in the space provided at the end of each line.

Part (b) [5 MARKS]

Compute the exact number of steps performed during lines 5–10. Show your work.

Part (c) [2 MARKS]

Let $T(n)$ represent the worst-case running time of PROD(A, B, n). Find a function $g : \mathbb{N} \to \mathbb{R}^{\geq 0}$ such that $T \in \Theta(g)$.

Question 3. (CONTINUED)

Part (d) [12 MARKS]

Prove that $T \in \Theta(g)$ for your function g from the previous part.

Question 4. [10 MARKS]

For each function f on the left, choose one expression $\mathcal{O}(g)$ on the right such that $f \in \mathcal{O}(g)$. Use each expression exactly once (*i.e.*, each expression must be used for some f and no expression can be used twice or more). No justification required.

a. $2^{n+7} \in$ ____________________	• $\mathcal{O}(1/n)$
b. $(n+2)(\frac{n-\log_2 n}{55}) \in$ ____________________	• $\mathcal{O}(1)$
c. $7 - 3/n \in$ ____________________	• $\mathcal{O}(\log n)$
d. $\frac{n+5}{2n^3+n^2} \in$ ____________________	• $\mathcal{O}(n)$
e. $2^{2n} \in$ ____________________	• $\mathcal{O}(n \log n)$
f. $3n^3 - n \in$ ____________________	• $\mathcal{O}(n^2)$
g. $(n+1)! \in$ ____________________	• $\mathcal{O}(n^{10})$
h. $4n + 3\log_2 n \in$ ____________________	• $\mathcal{O}(2^n)$
i. $\log_2(128n) \in$ ____________________	• $\mathcal{O}(10^n)$
j. $n\log_2(n^2+1) \in$ ____________________	• (none of the above)

Question 5. [20 MARKS]

For any function $g : \mathbb{N} \to \mathbb{R}^{\geq 0}$, define a function $\overline{g} : \mathbb{N} \to \mathbb{R}^{\geq 0}$ as follows:

$$\forall n \in \mathbb{N}, \overline{g}(n) = \lceil g(n) \rceil.$$

Prove or disprove each of the following statements.

Part (a) [8 MARKS]

For all functions $f : \mathbb{N} \to \mathbb{R}^{\geq 0}$ and $g : \mathbb{N} \to \mathbb{R}^{\geq 0}$, $f \in \mathcal{O}(g) \to f \in \mathcal{O}(\overline{g})$.

Question 5. (CONTINUED)

Part (b) [12 MARKS]

For all functions $f : \mathbb{N} \to \mathbb{R}^{\geq 0}$ and $g : \mathbb{N} \to \mathbb{R}^{\geq 0}$, $f \in \mathcal{O}(\overline{g}) \to f \in \mathcal{O}(g)$.

Bonus. [3 MARKS]

Write your student number where indicated at the bottom of every odd page, except for page 1.
Also, if you have not done so already, complete the identification section at the top of page 1.

Question 6. [10 MARKS]

Consider the following algorithm, where $x \in \mathbb{N}$:

while $x > 0$:
  if x is even:
    $x := x/2$
  else if $x > 5$:
    $x := x - 6$
  else:
    $x := x + 3$

For any iteration of the loop, let x_0 represent the value of x at the start of the iteration, and x_1 represent the value of x at the end of the iteration.

Part (a) [8 MARKS]

Prove that for every iteration of the loop, $x_1 > 0$. (HINT: if there is an iteration of the loop, what do we know about the value of x_0?)

Part (b) [2 MARKS]

Assuming that $x_1 > 0$ for every iteration of the loop, what can we conclude about this loop? (You can answer this part even if you did not answer the previous one.)

Question 7. [35 MARKS]

Part (a) [4 MARKS]

Perform the addition $(1101110)_2 + (10111)_2$ completely in binary, and show your work. (You will get *no* marks if you do not show your work.)

Part (b) [3 MARKS]

Briefly explain the difference between rounding and truncation. (HINT: What does each one apply to?)

Part (c) [5 MARKS]

Give an example to show that addition of floating-point numbers is not necessarily associative.
Give an example to show that multiplication of floating-point numbers is not necessarily associative.

Question 7. (CONTINUED)

Part (d) [4 MARKS]

We know that for numbers $x > 0, y > 0$, $\ln(x) - \ln(y) = \ln(x/y)$. Can this be used to compute the value of $\ln(x) - \ln(y)$ more accurately when x is very close to y, by avoiding catastrophic cancellation? (HINT: where is the ln function "sensitive"?)

Part (e) [4 MARKS]

Suppose that $a > 0$ is an approximation of some true value $t > 0$ with relative error r. Show that $a = (1 \pm r)t$.

Part (f) [3 MARKS]

Find the smallest values of t and $e_{\max}$ and the largest value of $e_{\min}$ such that the normalized floating-point system defined by $\beta = 10$, t, $e_{\min}$, $e_{\max}$ can represent the two numbers 3341.4 and 0.0000992 exactly.

Question 7. (CONTINUED)

Part (g) [3 MARKS]

Compute the absolute and relative error when π is represented as 3.14.

Part (h) [9 MARKS]

In the floating-point system with $\beta = 10$, $t = 2$, $e_{\min} = -3$, $e_{\max} = 1$, consider using the algorithm on the right to compute the value of

$$\underbrace{0.04 + 0.04 + \cdots + 0.04}_{n \text{ times}}$$

$t := 0.0 \times 10^0$
$i := 0$
while $i < n$:
    $t := t + 4.0 \times 10^{-2}$
    $i := i + 1$

Is this algorithm stable? Explain briefly. If the algorithm is not stable, explain how to perform the computation differently in a manner that is more stable. (You do not have to write a detailed algorithm.)

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Total Marks = 150