CSC165
Θ AND Ω

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To save use some typing, let's define F to be the set of functions from $N \to R^{\geq 0}$.

Consider $f \in F$ defined by $f(n) = 3n^2 + 2$. For $n \geq 2$, $3n^2 \leq f(n) \leq 4n^2$ (we saw the second inequality in the big-O notes). In other words, for large enough n, $f(n)$ is between two constant multiples of n^2. So for many computing purposes, $f(n)$ is essentially the same as n^2. Notice also that for $n \geq 2$, $\frac{1}{4}f(n) \leq n^2 \leq \frac{1}{3}f(n)$, so for large enough n, n^2 is between two constant multiples of $f(n)$.

We make the following definition:

For $f \in F, \Theta(f) = \left\{ g \in F : \exists c_1 \in R^+, \exists c_2 \in R^+, \exists b \in N, \forall n \in N, n \geq b \to c_1 f(n) \leq g(n) \leq c_2 f(n) \right\}$.

Informally, $\Theta(f)$ is the set of functions that grow at the same rate as f. We saw above that $3n^2 + 2 \in \Theta(n^2)$, and $n^2 \in \Theta(3n^2 + 2)$. This symmetry is true in general:

Theorem. $\forall f \in F, \forall g \in F, f \in \Theta(g) \leftrightarrow g \in \Theta(f)$.

Proof. Exercise. □

We can characterize Θ in terms of O:

Theorem. $\forall f \in F, \forall g \in F, f \in \Theta(g) \leftrightarrow (g \in O(f) \wedge f \in O(g))$.

Proof. Exercise. □

So, for example, $\log n \in O(n)$ but $n \notin O(\log n)$, so $\log n \notin \Theta(n)$.

The symmetry in these theorems supports our view of Θ as a kind of equivalnce. Furthermore, if two functions are equivalent in the sense of Θ, they behave the same way with respect to O:

Theorem. $\forall f \in F, \forall g \in F, f \in \Theta(g) \to \forall h \in F, (h \in O(f) \leftrightarrow h \in O(g)) \wedge (f \in O(h) \leftrightarrow g \in O(h))$.

Proof. Exercise. □

So, for example, any O results for n^2 are true for $3n^2 + 2$, and vice-versa.

For completeness, we mention:

Theorem. $\forall f \in F, f \in \Theta(f)$.

Proof. Exercise. □

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If we consider $f \in O(g)$ to mean f is as good as g, and $f \in \Theta(g)$ to mean f is *exactly* as good/bad as g, we might want a definition to capture 'f is as bad as g':

$$\text{For } f \in F, \Omega(f) = \left\{g \in F : \exists c \in R^{+}, \exists b \in N, \forall n \in N, n \geq b \rightarrow cf(n) \leq g(n)\right\}.$$

We leave it as an exercise to state results for Ω similar to the results for O.

It remains to compare Ω with O and Θ. Looking at the definitions, O and Ω look like two halves of Θ:

Theorem. $\forall f \in F, \forall g \in F, f \in \Theta(g) \leftrightarrow (f \in \Omega(g) \wedge f \in O(g))$.

Proof. Exercise. □

If f grows no faster than g, then we expect that g grows as fast as f, and vice versa:

Theorem. $\forall f \in F, \forall g \in F, f \in O(g) \leftrightarrow g \in \Omega(f)$.

We'll prove this one.

First we practice: suppose $f \in O(g)$ because $f(n) \leq 3g(n)$ for all $n \geq 40$. Then $\frac{1}{3}f(n) \leq g(n)$ for all $n \geq 40$, so $g \in \Omega(f)$. In general, the reciprocal of the c that shows $f \in O(g)$ shows $g \in \Omega(f)$, and the b requires no change.

Proof. Let $f \in F$, $g \in F$.

Suppose $f \in O(g)$.
  Then $\exists c \in R^{+}, \exists b \in N, \forall n \in N, n \geq b \rightarrow f(n) \leq cg(n)$.
  Let $c_O \in R^{+}$, $b_O \in N$ be such that $\forall n \in N, n \geq b_O \rightarrow f(n) \leq c_O g(n)$.
  Let $c = 1/c_O$. Then $c \in R^{+}$ since $c_O \in R^{+}$.
  Let $b = b_O$. Then $b \in N$ since $b_O \in N$.
    Let $n \in N$.
    Suppose $n \geq b$.
      Then $n \geq b_O$, so $f(n) \leq c_O g(n) = (1/c) g(n)$.
      Multiplying by c we get $cf(n) \leq g(n)$.
    Thus $n \geq b \rightarrow cf(n) \leq g(n)$.
    Since $n \in N$ is arbitrary: $\forall n \in N, n \geq b \rightarrow cf(n) \leq g(n)$.
  Since $c \in R^{+}$ and $b \in N$: $\exists c \in R^{+}, \exists b \in N, \forall n \in N, n \geq b \rightarrow cf(n) \leq g(n)$.
  Thus $g \in \Omega(f)$.
Therefore $f \in O(g) \rightarrow g \in \Omega(f)$.
Now suppose $g \in \Omega(f)$.
  [Exercise]
Therefore $g \in \Omega(f) \rightarrow f \in O(g)$.

Therefore $f \in O(g) \leftrightarrow g \in \Omega(f)$.

Since $f \in F$ and $g \in F$ are aribtrary: $\forall f \in F, \forall g \in F, f \in O(g) \leftrightarrow g \in \Omega(f)$. □