CSC165
DIRECT PROOF STRUCTURE

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For Universal Statements

The general structure of a direct proof of $\forall x \in D, p(x) \rightarrow q(x)$ is:

> Let $x \in D$.
> Suppose $p(x)$.
>   ___ [here's where we prove $q(x)$]
>   Thus $q(x)$.
> Hence $p(x) \rightarrow q(x)$.
> Since x is an arbitrary element of D: $\forall x \in D, p(x) \rightarrow q(x)$.

For example, the direct proof structure of $\forall x \in R, x > 0 \rightarrow \frac{1}{x+2} < 3$ is:

> Let $x \in R$.
> Suppose $x > 0$.
>   ___
>   Thus $\frac{1}{x+2} < 3$.
> Hence $x > 0 \rightarrow \frac{1}{x+2} < 3$.
> Since x is an arbitrary element of R: $\forall x \in R$, $x > 0 \rightarrow \frac{1}{x+2} < 3$.

A full proof, not just the structure, would fill in the "___":

> $x + 2 > 2$, since $x > 0$.
> $\frac{1}{x+2} < \frac{1}{2}$, since $x + 2 > 2$ and $2 > 0$.
> So $\frac{1}{x+2} < 3$, since $\frac{1}{x+2} < \frac{1}{2}$ and $\frac{1}{2} < 3$.

Exercise 1. Is the converse of the statement we proved true?

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Let's look at the outline in more detail.

We want to prove something for all $x \in R$. We could begin the outline with "For all $x \in R$:", but people traditionally like to think of working on one element at a time. Other common phrasings use "suppose", "for", or "if" instead of "let". These are all ways of stating the same *assumption.* The form of the statement we're proving allows this particular assumption.

The form of the statement also allows us to assume $x > 0$. An assumption may be used *later* in the proof, to justify our *conclusions.*

The "___" indicates where we would use the assumptions to eventually conclude that $\frac{1}{x+2} < 3$. Some words that indicate a conclusion are "thus", "so", "then", "hence", and "therefore".

Under the assumption that $x > 0$ we conclude $\frac{1}{x+2} < 3$, so we can conclude $x > 0 \rightarrow \frac{1}{x+2} < 3$. The indentation helps to highlight where we're using $x > 0$, and now that we have $x > 0 \rightarrow \frac{1}{x+2} < 3$ we won't be using $x > 0$ anymore.

We concluded $x > 0 \rightarrow \frac{1}{x+2} < 3$ based *only* on the assumption that $x \in R$. People indicate this by saying x is an *arbitrary* element of R. The word "since" is used to *remind* us of an earlier assumption or conclusion we're using now; "because" is another common word used when reminding. The final conclusion from this is that $x > 0 \rightarrow \frac{1}{x+2} < 3$ for any $x \in R$, in other words: $\forall x \in R, x > 0 \rightarrow \frac{1}{x+2} < 3$.

Consider now the full proof: it fills in the proof of $\frac{1}{x+2} < 3$. Just like the outline, it has conclusions based on earlier assumptions and conclusions, and it has reminders. Some of the reminders, e.g. $0 < 2$, are of conclusions of proofs made (a long time) before this proof. In general we may introduce conclusions proven before. Notice that we used some of these implicitly.

Exercise 2. Where did we use the known result that

$$\forall a \in R, \forall b \in R, \forall c \in R, (a < b \wedge b < c) \rightarrow a < c?$$

Let's summarize what we see in a direct proof:

(1) Assumptions: from the form of the statement.
(2) Conclusions: from earlier assumptions and conclusions in the proof, and conclusions proven elsewhere.
(3) Reminders: of assumptions and conclusions used in (2).

Not all universals are of an implication. If our universal statement is just $\forall x \in R, q(x)$ the direct proof structure is simply:

> Let $x \in D$.
>
> > ___ [proof of $q(x)$]
> >
> > Thus $q(x)$.
>
> Since x is an arbitrary element of D: $\forall x \in D, q(x)$.

The proof of $q(x)$ is likely to depend on previously proven results about D.

For Existential Statements

Consider the statement:

$$\exists x \in R, x^3 + 2x^2 + 3x + 4 = 2.$$

Here's a direct proof of it:

Let $x = -1$.
Then $x \in R$.
  Also,

$$\begin{aligned} & x^3 + 2x^2 + 3x + 4 \\ = \ & (-1)^3 + 2(-1)^2 + 3(-1) + 4 \\ = \ & -1 + 2 - 3 + 4 \\ = \ & 2. \end{aligned}$$

(Note that the final four lines are really three sentences written in a special form.)

The general structure of a direct proof of $\exists x \in D, q(x)$ is:

Let $x =$___. [we get to pick a specific value, unlike for universal]
___, so $x \in D$. [but we have to prove it's in D]
  ___ [proof of $q(x)$]
  Thus $q(x)$.
Since $x \in D$ and $q(x)$: $\exists x \in D, q(x)$.

We put the proof that $x \in D$ on one line above, because it's often obvious that $x \in D$ and we don't even fill in the "___". But there are times where some justification is in order, and that justification can require multiple lines.

Exercise 3. What are the assumptions, conclusions and reminders in our example proof and the outline?

A common form of existential is $\exists x \in D, p(x) \wedge q(x)$. Instead of looking at the effect of $\wedge$ here, we'll discuss it during one of the examples in the next section.

With Multiple Quantifiers

Consider

$$(S1)\ \forall x \in I, \exists y \in I, y > x.$$
$$(S2)\ \exists y \in I, \forall x \in I, y > x.$$

Since (S1) is a universal, we begin with:

Let $x \in I$.
  ___
  Thus $\exists y \in I, y > x$.
Since x is an arbitrary element of I: $\forall x \in I, \exists y \in I, y > x$.

We fill in "___" with the structure for proving the existential $\exists y \in I, y > x$:

Let $x \in I$.
  Let $y =$___.
  ___, so $y \in I$.
    ___
    Thus $y > x$.
  Since $y \in I$ and $y > x$: $\exists y \in I, y > x$.
Since x is an arbitrary element of I: $\forall x \in I, \exists y \in I, y > x$.

Multiple quantifiers lead to nesting in the proof structure.

Statement (S1) is true. Let's see a proof of it using this structure.

Let $x \in I$.
  Let $y = x + 1$.
  Then $y \in I$.
    Since $1 > 0$, we get $y = x + 1 > x + 0 = x$.
    Thus $y > x$.
  Since $y \in I$ and $y > x$: $\exists y \in I, y > x$.
Since x is an arbitrary element of I: $\forall x \in I, \exists y \in I, y > x$.

Because x is introduced before y we may use x, and the assumption that it's in I, to define y.

The structure for a direct proof of (S2) is generated similarly:

Let $y =$___.
___, so $y \in I$
  Let $x \in I$.
    ___
    Thus $y > x$.
  Since x is an arbitrary element of I: $\forall x \in I, y > x$.
Since $y \in I$ and $\forall x \in I, y > x$: $\exists y \in I, \forall x \in I, y > x$.

This is similar to the outline for (S1). The crucial difference is that we must choose y before x is introduced, so y can't depend on x.

Since (S2) is false, it's not possible to fill in the blanks and get a correct proof. To prove it's false, we can't just say we weren't able to fill in the blanks (maybe we weren't clever enough!). Instead, we would prove its negation is true.

Exercise 4. Write out the negation of (S2) and the direct proof structure for the negation (after moving the $\neg$ inside).

Our final example is for a statement many of you are familiar with:

$$\forall \epsilon \in R, \epsilon > 0 \rightarrow \exists \delta \in R, \delta > 0 \wedge \forall x \in R, 0 < |x-a| < \delta \rightarrow |f(x) - l| < \epsilon.$$

The direct proof structure for it is:

Let $\epsilon \in R$.
Suppose $\epsilon > 0$.
Let $\delta =$ ___.
___, so $\delta \in R$. [now to prove the $\wedge$, we prove each part]

___
Hence $\delta > 0$. [proved first part of the $\wedge$]
Let $x \in R$.
Suppose $0 < |x-a| < \delta$. [an implicit $\wedge$ – may conclude both parts]
Then $0 < |x-a|$. [from first half of the $\wedge$]
Also, $|x-a| < \delta$. [from second half]

___
Thus $|f(x) - l| < \epsilon$.
Hence $0 < |x-a| < \delta \rightarrow |f(x) - l| < \epsilon$. [proved second part of the $\wedge$]
Since x is an arbitrary element of R: $\forall x \in R, 0 < |x-a| < \delta \rightarrow |f(x) - l| < \epsilon$.
Thus $\delta > 0 \wedge \forall x \in R, 0 < |x-a| < \delta \rightarrow |f(x) - l| < \epsilon$.
Since $\delta \in R$ and $\delta > 0 \wedge \forall x \in R, 0 < |x-a| < \delta \rightarrow |f(x) - l| < \epsilon$:

$$\exists \delta \in R, \delta > 0 \wedge \forall x \in R, 0 < |x-a| < \delta \rightarrow |f(x) - l| < \epsilon$$

Since ϵ is an arbitrary element of R:

$$\forall \epsilon \in R, \epsilon > 0 \rightarrow \exists \delta \in R, \delta > 0 \wedge \forall x \in R, 0 < |x-a| < \delta \rightarrow |f(x) - l| < \epsilon.$$

This examples shows both how to conclude $p \wedge q$, and how to use $p \wedge q$. Once we have p and q, then we can conclude $p \wedge q$. If we have $p \wedge q$, we can conclude p and we can conclude q. Unfortunately, $p \vee q$ is not so simple.