CSC 165

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Existential Quantification

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(2) becomes:

Some employee makes more than 100K.

Are (1) and (2) true?

Recall: (1) There is an employee who makes less than 25K.
Is (1) true for Buffy?

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What does (2) mean in terms of sets?

Note: We can't disprove it with a single counterexample.

Exists: Symbolically

Symbol: $\exists$ "there exists"

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Rewrite (1):

- ∃ employee, the employee makes less than 25K
- ∃ employee e, e makes less than 25K
- Let E = the set of employees
  Let L(e) = e makes less than 25K
  ∃ e in E, L(e)

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  $\exists$ e in E, L(e)

(From least to most preferred.)

Exists: Everday Language

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- a / an / some / at least one
- such that / for which

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Note: “some” is inclusive. “some object is a P” is true if “every object is a P” is true.

When is ∃ false?

When is ∃ x, P(x) false?

When is $\exists$ false?

When is $\exists$ x, P(x) false?

When P(x) is false for every x in the domain.

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Saying “Some employee makes more than 100K” is false is equivalent to saying:

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When is $\exists$ x, P(x) false?

When P(x) is false for every x in the domain.

In other words, $\neg\exists$ x, P(x) is equivalent to $\forall$ x, $\neg$ P(x).

Saying "Some employee makes more than 100K" is false is equivalent to saying:

Every employee does not make more than 100K.

Restrict Domain

Let's restrict the domain further:

(3) Some female employee makes less than 25K.
(4) There exists a male employee making less than 25K.

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For $\exists$, the restriction is expressed using "and".

In general,

- "Every P is a Q" becomes $\forall$ x, P(x) $\rightarrow$ Q(x)
- "Some P is a Q" becomes $\exists$ x, P(x) and Q(x)

Example

For example, (3) becomes:

$\exists$ employee e, e is female and e makes less than 25K.

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Buffy is female and Buffy makes less than 25K.

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(3) is true because:
Buffy is female and Buffy makes less than 25K.

(4) is false because:
Each male employee does't make less than 25K.

($\forall$ employees e, e male $\rightarrow \neg$ makes less than 25K)

In General

- “There is no P that is a Q” is equivalent to “every P is not a Q”

$$\neg(\exists x, P(x) \text{and} Q(x)) \leftrightarrow \forall x, (P(x) \rightarrow \neg Q(x))$$

In General

- "There is no P that is a Q" is equivalent to "every P is not a Q"

$$\neg(\exists x, P(x) \text{and} Q(x)) \leftrightarrow \forall x, (P(x) \rightarrow \neg Q(x))$$

- "not every P is a Q" is equivalent to "there is some P that is not a Q"

$$\neg(\forall x, P(x) \rightarrow Q(x)) \leftrightarrow \exists x, (P(x) \text{and} \neg Q(x))$$

Repeated Quantifiers

Many useful properties apply to multiple elements. For example:

Let E = employees
Let I = integers
For e in E, k in I, let sl(e, k) = e makes salary less than k
For e in E, let f(e) = e is female

Repeated Quantifiers

Many useful properties apply to multiple elements. For example:

Let E = employees
Let I = integers
For e in E, k in I, let sl(e, k) = e makes salary less than k
For e in E, let f(e) = e is female

Now we can rewrite (3) as:

$\exists$ x in E, f(x) and sl(x, 25000)

Repeated Quantifiers (continued)

Even better: use “less than” to make expressions more flexible.

For e in E, k in I, let s(e, k) = e makes salary k

Repeated Quantifiers (continued)

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For e in E, k in I, let s(e, k) = e makes salary k

Rewrite(3): $\exists$ k $\in$ I, $\exists$ x $\in$ E, f(x) and s(x, k) and k < 25000

There are 3 other equivalent versions.

And

Use “and” to make a sentence that claims two sentences are both true.

For example,

(5) The employee makes less than 75K and more than 25K.

Is (5) true for Anya?

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For example,

(5) The employee makes less than 75K and more than 25K.

Is (5) true for Anya?

Yes, Anya makes less than 75K and Anya makes more than 25K.

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For example,

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Is (5) true for Anya?

Yes, Anya makes less than 75K and Anya makes more than 25K.

Is (5) true for Buffy?

And

Use "and" to make a sentence that claims two sentences are both true.

For example,

(5) The employee makes less than 75K and more than 25K.

Is (5) true for Anya?

Yes, Anya makes less than 75K and Anya makes more than 25K.

Is (5) true for Buffy?

No, Buffy does not make more than 75K.

And: Symbolically and Everday

Symbol: $\wedge$ “and”
Viewing sentences as sets, $\wedge$ is the intersection.

In everyday language, “and” is used to:

- group things, and
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I have a pen and a telephone. (group things)
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In everyday language, “and” is used to:

- group things, and
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For example,
I have a pen and a telephone. (group things)
I have something which is a pen and a telephone. (join sentences)

We only use $\wedge$ for joining sentences.

Or

Use “or” to make a sentence that claims at least one of two sentences is true.

For example,

(6) The employee is female or (the employee) makes less than 45K.

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Yes, because Anya is female.

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Or

Use "or" to make a sentence that claims at least one of two sentences is true.

For example,

(6) The employee is female or (the employee) makes less than 45K.

Is (6) true for Anya?

Yes, because Anya is female.

Is (6) true for Buffy?

Yes, because Buffy is female. Also true because Buffy makes less than 45K.

Or (continued)

For example,

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Is (6) true for Xander?

Or (continued)

For example,

(6) The employee is female or (the employee) makes less than 45K.

Is (6) true for Xander?

No, because he is not female and he does not make less than 45K.

In terms of sets, “or” is the union of two sets.

Or (continued)

For example,

(6) The employee is female or (the employee) makes less than 45K.

Is (6) true for Xander?

No, because he is not female and he does not make less than 45K.

In terms of sets, “or” is the union of two sets.

Note: For us, “or” is inclusive, because it includes the case where more than one of the properties is true.

DeMorgan's Laws

- A sentence "s1 and s2" is false exactly when:

DeMorgan's Laws

- A sentence "s1 and s2" is false exactly when:

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Symbolically: $\neg(s1 \land s2) \leftrightarrow (\neg s1) \lor (\neg s2)$

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Mixed Quantifiers

Consider the table, which shows who repects who (a * in row x and column y means x respects y):

	A	B	D	G	W	X
A	*					
B		*	*	*	*	*
D		*	*	*	*	*
G		*	*	*	*	
W		*	*	*		
X		*	*	*	*	

Let r(x, y) = x respects y

Mixed Quantifiers: Example

(7) y is respected by someone

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Is (7) true for y = A?

Mixed Quantifiers: Example

(7) y is respected by someone

Symbolically: $\exists x \in P, r(x, y)$

Is (7) true for y = A?
Yes (look at the A column)

Mixed Quantifiers: Example

(7) y is respected by someone

Symbolically: $\exists x \in P, r(x, y)$

Is (7) true for y = A?
Yes (look at the A column)

Is (7) true for y = A, y = B, ... , y = X?

Mixed Quantifiers: Example

(7) y is respected by someone

Symbolically: $\exists x \in P, r(x, y)$

Is (7) true for y = A?
Yes (look at the A column)

Is (7) true for y = A, y = B, ... , y = X?
Yes! So we can write: $\forall y \in P, \exists x \in P, r(x, y)$

Mixed Quantifiers: Example

(7) y is respected by someone

Symbolically: $\exists x \in P, r(x, y)$

Is (7) true for y = A?
Yes (look at the A column)

Is (7) true for y = A, y = B, ... , y = X?
Yes! So we can write: $\forall y \in P, \exists x \in P, r(x, y)$

In English: Everyone has someone who respects them.

Mixed Quantifiers: Example

A row x is fulls of *s:

$\forall y \in P, r(x, y)$

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In English: someone who respects everyone

Mixed Quantifiers: Example

A row x is fulls of *s:
$\forall y \in P, r(x, y)$

$\exists x \in P, \forall y \in P, r(x, y)$

In English: someone who respects everyone
This is false. No row is full of *s.

Mixed Quantifiers: Example

A row x is fulls of *s:
$\forall y \in P, r(x, y)$

$\exists x \in P, \forall y \in P, r(x, y)$

In English: someone who respects everyone
This is false. No row is full of *s.

The difference between (8) and (9) is the order of the quantifiers.

Existential quantifiers involve choice. The farther to the right an existential quantifier is, then more that we can tailor our choice.