

## Sentences and Statements

Consider:

The employee makes less than 55K.  
Every employee makes less than 55K.

(1) depends on specifying the employee:

- for Anya, it's false
- for Buffy, it's true

(2) does not depend on a particular employee, so we can ask whether it is true (in fact, it is false).

Precise notation and terminology:

Let  $l(x)$  denote "employee  $x$  makes less than 55K".

We can write (2) as:

for all employees  $x$ ,  $l(x)$ .

(2) is called a statement. It does not refer to any unspecified objects (unquantified variables), and it is true or false (but not both).

(1) is called a sentence. It may refer to unspecified objects, and when objects are specified (by substituting for the variable(s)), it is true or false (but not both).

A sentence does not require referring to unspecified objects. Every statement is a sentence. A sentence that refers to unspecified objects is an "open sentence" or "open statement".

Therefore: a sentence is a statement  $\iff \neg$  (the sentence is an open statement).

Notation:

- $\iff$  - if and only if
- $\neg$  - not

Universal quantification can turn a sentence into a statement. For example, (2) is the universal quantification of (1).

## Symbolic notation for universal quantification

Universal quantification can be written symbolically as:  $\forall$ .

The symbol  $\forall$  is read as "for all".

We must be clear about the set of objects to which the sentence can refer (called the "domain").

For example, (2) can be written as:

$\forall$  employees, the employee makes less than 55K.

An even better option is to introduce the name for the unspecified object(s):

$\forall$  employees  $e$ ,  $e$  makes less than 55K.

The best option, is to introduce the domain explicitly and use symbolic notation for the sentence that will be quantified:

Let  $E$  = set of employees

Let  $l(e)$  = employee  $e$  makes less than 55000

$\forall e \in E, l(e)$

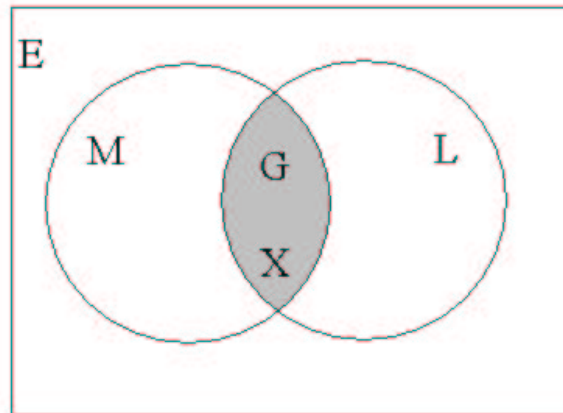
### The Implications of Implication

Suppose we know “every P is a Q” is true. Going back to our database example:

(3) If an employee is male, then he makes less than 55000.

Consider (3) again in terms of sets:

- E: Employees
- M: Male employees
- L: Employees making less than 55000



(3) tells us about male employees. What does it tell us about Xander?

Does (3) tell us anything about an employee who makes less than 55000, if that’s all we know about the employee?

About Dawn, who makes less than 55000?  
About Xander, who makes less than 55000?

Does (3) tell us anything about each of the non-male employees?

About the female Dawn?  
About the female Anya?

Does it tell us about each employee not making less than 55000?

About Anya, who doesn’t make less than 55000?

Knowing that P implies Q doesn’t tell us anything more about something that we know is :

not a P  
a Q

It does tell us more if we just know that something is:

a P (then we know it’s a Q)  
not a Q (then we know it’s not a P)

In general, “P implies Q” is false exactly when P is true and Q is false (it is true in all other cases).

Symbolic notation for implication

Implication written symbolically as:  $\rightarrow$ .

The symbol  $\rightarrow$  is read as “implies”.

So “P implies Q” is written as:  $P \rightarrow Q$ . For example, (3) becomes:

employee is male  $\rightarrow$  employee makes less than 55K

Implication and Universal quantification

Consider:

(4) Every male employee makes less than 55000.

The universal quantification is explicit here (“Every”).

Often, the universal quantification is implicit:

(4') If an employee is male then the employee makes less than 55000.

The indefinite article “an” (as opposed to “the”) usually indicates this.

In (4') the implication is explicit, but in (4) it's implicit.

We can make both explicit:

(4'') For every employee, if the employee is male then the employee makes less than 55000.

(Notice the switch to “the”)

The statements (4), (4') and (4'') all mean the same thing, and symbolically we write them all the same way with the universal quantification and the implication explicit:

Let  $E$  and  $l$  be as we defined earlier.

Let  $m(e)$  = employee  $e$  makes less than 55000.

$\forall e \in E, m(e) \rightarrow l(e)$ .

Contrapositive

The contrapositive of  $P \rightarrow Q$  is  $\neg Q \rightarrow \neg P$ .

In English, the contrapositive of “P implies Q” is “all non-Qs are non-Ps”.

For example, the contrapositive of (3) is:

Employee doesn't make less than 55000  $\rightarrow$  employee is not male

or, in other words:

Employee makes at least 55000  $\rightarrow$  employee is female

Does the contrapositive of (3) tell us everything that (3) does?

Compare their Venn diagrams. The Venn diagrams are the same, so they are equivalent.

What is the contrapositive of a contrapositive?

The contrapositive of  $P \rightarrow Q$  is  $\neg Q \rightarrow \neg P$ .

The contrapositive of  $\neg Q \rightarrow \neg P$  is  $\neg\neg P \rightarrow \neg\neg Q$ , which is equivalent to  $P \rightarrow Q$ .

### Converse

The converse of  $P \rightarrow Q$  is  $Q \rightarrow P$ .

In other words, the converse of “P implies Q” is “Q implies P”.

Is a sentence equivalent to its converse? Once again, consider the Venn diagram.

So the converse says something different than the original statement.

To summarize, consider the example:

$$(5) \quad x = 0 \rightarrow xy = 0$$

If we know  $x = 0$ , then we also know  $xy = 0$ .

If we know  $x \neq 0$ , then we don't know anything about  $xy$ .

If we know  $xy = 0$ , then we don't know anything about  $x$ .

If we know  $xy \neq 0$ , then we know  $x \neq 0$ .

The contrapositive of (5) is:

$$xy \neq 0 \rightarrow x \neq 0$$

It is equivalent to (5).

The converse of (5) is:

$$xy = 0 \rightarrow x = 0$$

It is equivalent to its own contrapositive:

$$x \neq 0 \rightarrow xy \neq 0$$

but, it is not equivalent to (5). In particular, (5) is true but the converse of (5) is not true.