

TO STUDENTS: This file contains the solutions to the midterm test together with the marking scheme and comments for each question. Please read the solutions and the marking schemes and comments carefully. Make sure that you understand why the solutions given here are correct, that you understand the mistakes that you made (if any), and that you understand *why* your mistakes were mistakes.

Remember that although you may not agree completely with the marking scheme given here it was followed the same way for all students. We will remark your test only if you can clearly demonstrate that the marking scheme was not followed correctly. We will make *no* exception to the marking scheme, unless you can clearly demonstrate that it is somehow incorrect.

For all remarking requests, please submit your request **in writing** directly to your instructor. For all other questions, please don't hesitate to ask your instructor during office hours or by e-mail.

TO MARKER: In general, having the right idea should count for approximately half of the marks even if it isn't expressed perfectly. Start from the assumption that students know what they are doing, unless they give you evidence to the contrary. In particular, a correct answer followed or preceded by an incorrect justification or explanation deserves fewer marks than the correct answer alone. In contrast, an incorrect answer accompanied by a correct explanation or followed by an acknowledgement that the answer is incorrect deserves more marks than the incorrect answer alone. In general, try not to be too picky on the details except when they are critical to the solution. The important goal is consistency: even if you feel your choice is a little lenient, or a little harsh, stick with it for everyone.

Remember that students should get 20% of the marks for a question (don't round) if they write "I do not know how to answer" or something similar, as long as they write nothing else (or that they cross off anything else they wrote to make it clear it should not be marked). If a student writes "I do now know how to answer" before or after an attempted solution, mark only the attempted solution and ignore the sentence stating that they do not know.

While you mark, please keep track of common student errors and how they were marked, as well as any interpretations of or minor modifications to the marking scheme (including any further breakdown of the marks that you decide to use), so that it is easy to figure out later how a question was marked. Also, please make note of how well each question or each part of a question was answered in general, and of any serious misconceptions or apparent gaps in student's knowledge that you noticed. (These comments will be typeset with the solutions and posted on the course website so that students can find out where and why they lost marks.)

Finally, please remember to give students enough feedback on their copy of the test so that they can easily figure out what they did wrong (if anything) from the solutions, marking schemes, and comments, together with your feedback. In particular, you may find it convenient to use codes (like "E1") to report common errors on a student's paper, and list the codes and their meaning in your marking comments.

### Question 1. [12 MARKS]

Consider the following sentences about sequences of natural numbers  $a_0, a_1, a_2, \dots$ :

$$(S1) \quad \forall i \in \mathbb{N} \exists j \in \mathbb{N}, j < i \wedge a_j = a_i$$

$$(S2) \quad \forall i \in \mathbb{N} \exists j \in \mathbb{N}, j > i \wedge a_j = a_i$$

$$(S3) \quad \forall i \in \mathbb{N} \exists j \in \mathbb{N}, j \neq i \wedge a_j = a_i$$

**Question 1.** (CONTINUED)**Part (a)** [3 MARKS]

Rewrite each of the sentences above in clear, natural English.

(S1): “Every element in the sequence is repeated at a smaller position.”

(S2): “Every element in the sequence is repeated at a larger position.”

(S3): “Every element in the sequence is repeated at a different position.”

SOLUTION: (See above.)

MARKING SCHEME:

- 1 for each correct answer
- $-1$  overall if the sentences are translated word-for-word and the English is very poor (*e.g.*, “for all  $i$  in the set of natural numbers, there exists  $j$  in the set of natural numbers,  $j$  less than  $i$  and  $a_j$  equal to  $a_i$ ”).

However, something like this is acceptable: “For all natural numbers  $i$ , there is some natural number  $j$  such that  $j < i$  and  $a_j = a_i$ .” (Because the English is at least grammatically correct and the meaning is clearly correct as well.)

COMMENTS: Poor or very confused English sentences were penalized by  $-0.5$  to  $-1.5$ , depending on the severity of the problem.

**Part (b)** [3 MARKS]

Determine whether each sentence above is true or false for the following sequence and circle the appropriate answers:

12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0, 0, 0, ...

(S1) is: TRUE /  FALSE / IMPOSSIBLE TO DETERMINE\* / I DON'T KNOW (FOR 20%)

(S2) is: TRUE /  FALSE / IMPOSSIBLE TO DETERMINE\* / I DON'T KNOW (FOR 20%)

(S3) is: TRUE /  FALSE / IMPOSSIBLE TO DETERMINE\* / I DON'T KNOW (FOR 20%)

SOLUTION: (See above.)

MARKING SCHEME:

- 1 for each correct answer (0.2 for each “I don’t know”)

**Part (c)** [3 MARKS]

Determine whether each sentence above is true or false for the following sequence and circle the appropriate answers:

0, 0, 1, 0, 1, 2, 0, 1, 2, 3, 0, 1, 2, 3, 4, 0, 1, 2, 3, 4, 5, ...

(S1) is: TRUE /  FALSE / IMPOSSIBLE TO DETERMINE\* / I DON'T KNOW (FOR 20%)

(S2) is:  TRUE / FALSE / IMPOSSIBLE TO DETERMINE\* / I DON'T KNOW (FOR 20%)

(S3) is:  TRUE / FALSE / IMPOSSIBLE TO DETERMINE\* / I DON'T KNOW (FOR 20%)

SOLUTION: (See above.)

MARKING SCHEME:

- 1 for each correct answer (0.2 for each “I don’t know”)

**Question 1.** (CONTINUED)**Part (d)** [3 MARKS]

Determine whether each sentence above is true or false for the following sequence and circle the appropriate answers:

0, 1, 0, 2, 1, 3, 2, 4, 3, 5, 4, 6, 5, 7, 6, 8, 7, 9, 8, 10, 9, ...

(S1) is: TRUE /  FALSE / IMPOSSIBLE TO DETERMINE\* / I DON'T KNOW (FOR 20%)

(S2) is: TRUE /  FALSE / IMPOSSIBLE TO DETERMINE\* / I DON'T KNOW (FOR 20%)

(S3) is:  TRUE / FALSE / IMPOSSIBLE TO DETERMINE\* / I DON'T KNOW (FOR 20%)

SOLUTION: (See above.)

MARKING SCHEME:

- 1 for each correct answer (0.2 for each “I don’t know”)

COMMENTS: Many students incorrectly stated that (S2) was true.

**Question 2.** [28 MARKS]

Let  $P$  represent the set of all programs.

Let  $T$  represent the set of all test cases.

Let  $f(p, t)$  represent the sentence: “program  $p$  failed test  $t$ ”.

Rewrite each of the following sentences symbolically, using only the domains and predicate defined above (in other words, you are *not* allowed to define new predicates in your answers). Of course, you can (and should) use quantifiers and connectives where they are appropriate.

GENERAL COMMENTS: Many students misused the notation in one way or another. Here are some of the more common errors.

- Unnecessary quantified variable, *e.g.*,  $\forall x \in P, f(r, w)$ . (Penalized  $-1$  overall.)
- “ $\exists$ ” instead of “ $\neg\exists$ ”. (Penalized  $-1$  overall.)
- Using  $P$  and  $T$  as if they were predicates, *e.g.*,  $\forall x, P(x) \rightarrow \dots$ . (Penalized  $-1$  overall.)
- Writing domain information for free variables, *e.g.*, “ $r \in P, w \in T, f(r, w)$ ”. This is not necessary because the definition of  $f(r, w)$  already includes the fact that  $r \in P$  and  $w \in T$ .  
(Not penalized, except when free variables follow quantified ones, *e.g.*, “ $\forall t \in T, r \in P, s \in P, \dots$ ”, because then it is unclear whether you mean  $\forall$  on all of the variables or not.)
- Using connectives between quantifiers, *e.g.*, “ $\exists x \in P \wedge \exists y \in P, \dots$ ”. This is incorrect because  $\wedge$  and  $\vee$  are only used to join *sentences*. (Not penalized.)

**Part (a)** [3 MARKS]

“Program  $r$  passed test  $w$ .”

SOLUTION:  $\neg f(r, w)$

MARKING SCHEME:

- 1 mark: free variables  $r, w$ ; no other free variable
- 1 mark: “ $\neg f(-, -)$ ”
- 1 mark: arguments  $r, w$

**Part (b)** [3 MARKS]

“Program  $q$  failed some test.”

SOLUTION:  $\exists t \in T, f(q, t)$

**Question 2.** (CONTINUED)

MARKING SCHEME:

- 1 mark: free variable  $q$ ; no other free variable
- 1 mark: “ $\exists t \in T$ ”
- 1 mark: “ $f(q, t)$ ”

**Part (c)** [3 MARKS]

“No program failed every test.”

SOLUTION:  $\neg \exists p \in P, \forall t \in T, f(p, t)$  (equivalently:  $\forall p \in P, \exists t \in T, \neg f(p, t)$ )

MARKING SCHEME:

- 1 mark: no free variable
- 1 mark: correct quantifiers
- 1 mark: correct predicate

**Part (d)** [3 MARKS]“Every program that passed test  $u$  failed test  $w$ .”SOLUTION:  $\forall p \in P, \neg f(p, u) \rightarrow f(p, w)$ 

MARKING SCHEME:

- 1 mark: free variables  $u, w$ ; no other free variable
- 1 mark: “ $\forall p \in P$ ”
- 1 mark: “passed  $u \rightarrow$  failed  $w$ ”

**Part (e)** [3 MARKS]“Programs  $r$  and  $s$  passed exactly the same tests.”SOLUTION:  $\forall t \in T, f(r, t) \leftrightarrow f(s, t)$ 

MARKING SCHEME:

- 1 mark: free variables  $r, s$ ; no other free variable
- 1 mark: “ $\forall t \in T$ ”
- 1 mark: “ $r$  passed  $\leftrightarrow$   $s$  passed” (or equivalent)

COMMENTS: A common mistake was to write  $\neg f(r, t) \wedge \neg f(s, t)$  instead of  $\neg f(r, t) \leftrightarrow \neg f(s, t)$ , and to use an existential quantifier ( $\exists t \in T$ ) instead of a universal quantifier.

**Part (f)** [3 MARKS]“Program  $r$  failed every test that was failed by some other program.”SOLUTION:  $\forall t \in T, (\exists p \in P, f(p, t)) \rightarrow f(r, t)$  (equivalently:  $\forall t \in T, \forall p \in P, f(p, t) \rightarrow f(r, t)$ )

MARKING SCHEME:

- 1 mark: “ $\forall t \in T$ ”
- 1 mark: “ $(\exists p \in P, f(p, t))$ ”
- 1 mark: “ $\rightarrow f(p, t)$ ”

COMMENTS: The formula  $\forall t \in T, \exists p \in P, f(p, t) \rightarrow f(r, t)$  is ambiguous because it would normally mean:  $\forall t \in T, \exists p \in P, (f(p, t) \rightarrow f(r, t))$ , whose meaning is quite different from the correct answer. However, I gave students the benefit of the doubt if they omitted the parentheses, as long as everything else was correct.

**Question 2.** (CONTINUED)**Part (g)** [10 MARKS]

Suppose that the sentences from parts (a) to (f) on the previous page are all true. For each sentence below, determine its value and circle the appropriate answer. Briefly justify each of your answers. (Note that it is possible to answer this part even if you did not complete parts (a) to (f).)

“ $r$  passed test  $w$ ” is:

TRUE / FALSE / IMPOSSIBLE TO DETERMINE\* / I DON'T KNOW (FOR 20%)

*Justification:*

SOLUTION: (See above for answer.)  
from (a).

MARKING SCHEME:

- 1 mark: answer
- 1 mark: justification

COMMENTS: Common error: “true because no statement contradicts it”. (By that reasoning, the statement “ $x$  passed test  $m$ ” should also be true, even though we know nothing about  $x$  or  $m$ .)

“ $r$  passed test  $u$ ” is:

TRUE /  FALSE / IMPOSSIBLE TO DETERMINE\* / I DON'T KNOW (FOR 20%)

*Justification:*

SOLUTION: (See above for answer.)  
from (a) and the contrapositive of (d)

MARKING SCHEME:

- 1 mark: answer
- 1 mark: justification

“ $s$  failed test  $w$ ” is:

TRUE /  FALSE / IMPOSSIBLE TO DETERMINE\* / I DON'T KNOW (FOR 20%)

*Justification:*

SOLUTION: (See above for answer.)  
By (a) and (e).

Alternatively, the contrapositive of (f) asserts that every test that  $r$  passed was passed by all programs. Together with (a), this implies that every program passed test  $w$ , including  $s$ .

MARKING SCHEME:

- 1 mark: answer
- 1 mark: justification

“all programs passed test  $w$ ” is:

TRUE / FALSE / IMPOSSIBLE TO DETERMINE\* / I DON'T KNOW (FOR 20%)

*Justification:*

SOLUTION: (See above for answer.)

The contrapositive of (f) asserts that every test that  $r$  passed was passed by all programs. Together with (a), this implies that every program passed test  $w$ .

**Question 2.** (CONTINUED)

MARKING SCHEME:

- 1 mark: answer
- 1 mark: justification

COMMENTS: Common errors: “(b) means  $q$  failed  $w$ ” (that’s not true: it does not say what test  $q$  failed); “(c) means all programs passed  $w$ ” (that’s not true: it does not say what test was passed for each program).

“some program failed test  $w$ ” is:

TRUE /  FALSE / IMPOSSIBLE TO DETERMINE\* / I DON’T KNOW (FOR 20%)

*Justification:*

SOLUTION: (See above for answer.)

From the second statement above,  $r$  failed  $u$ .

Alternatively, from the contrapositive of (d) and the previous statement, every program passed test  $u$ .

MARKING SCHEME:

- 1 mark: answer
- 1 mark: justification

**Question 3.** [10 MARKS]

Consider the following sentence:

( $S$ )  $p$  and  $q$  are both true only if  $r$  is true.

**Part (a)** [2 MARKS]

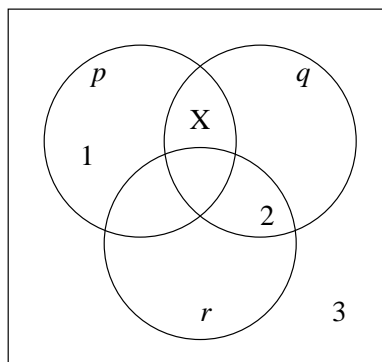
Rewrite ( $S$ ) using symbolic notation.

SOLUTION:  $(p \wedge q) \rightarrow r$

MARKING SCHEME:

- 1 for having  $(p \wedge q)$ ,  $\rightarrow$ , and  $r$
- 1 for having  $\rightarrow$  in the correct direction

Next, consider the following diagram illustrating the possible combinations of values for  $p, q, r$ :



**Question 3.** (CONTINUED)**Part (b)** [1 MARK]

Is ( $S$ ) true or false for the region numbered 1 in the diagram above (circle the appropriate answer)?

TRUE / FALSE / IMPOSSIBLE TO DETERMINE\* / I DON'T KNOW (FOR 20%)

SOLUTION: (See above.)

MARKING SCHEME:

- 1 for each correct answer (0.2 for each “I don’t know”)

**Part (c)** [1 MARK]

Is ( $S$ ) true or false for the region numbered 2 in the diagram above (circle the appropriate answer)?

TRUE / FALSE / IMPOSSIBLE TO DETERMINE\* / I DON'T KNOW (FOR 20%)

SOLUTION: (See above.)

MARKING SCHEME:

- 1 for each correct answer (0.2 for each “I don’t know”)

**Part (d)** [1 MARK]

Is ( $S$ ) true or false for the region numbered 3 in the diagram above (circle the appropriate answer)?

TRUE / FALSE / IMPOSSIBLE TO DETERMINE\* / I DON'T KNOW (FOR 20%)

SOLUTION: (See above.)

MARKING SCHEME:

- 1 for each correct answer (0.2 for each “I don’t know”)

**Part (e)** [5 MARKS]

In the diagram above, put an “X” in every region where sentence ( $S$ ) is false.

SOLUTION: (See diagram above.)

MARKING SCHEME:

- 1 for each region other than 1, 2, and 3 either correctly having an “X” or correctly not having an “X”

**Bonus.** [1 MARK]

Write your student number where indicated at the bottom of every odd page, except for page 1.

Also, if you have not done so already, complete the identification section at the top of page 1.

MARKING SCHEME:  $-1$  for each odd page where the student number is missing, or for writing the student number at the bottom of page 1, or for each line of information missing from the identification section at the top of page 1

COMMENTS: This was a “free” mark (you did not need to know any material to get it), but you had to follow the directions *precisely* in order to get it!