

1. Recall from A2, the following three sentences about a sequence of natural numbers  $a_0, a_1, a_2, \dots$ :

$$(S1) \quad \forall i \in \mathbb{N}, a_i \leq a_{i+1} \leq a_{i+2} \vee a_i \geq a_{i+1} \geq a_{i+2}$$

$$(S2) \quad \forall i \in \mathbb{N}, \exists j \in \mathbb{N}, \forall k \in \mathbb{N}, k \geq j \rightarrow a_i \leq a_k$$

$$(S3) \quad \exists i \in \mathbb{N}, \forall j \in \mathbb{N}, \exists k \in \mathbb{N}, k \geq j \wedge a_i < a_k$$

Recall the sequence of integers:

$$(A) \quad -1, 0, 1, 0, -1, 0, 1, 0, -1, \dots$$

(A) can also be defined as:

$$\text{For } n = 0, 1, 2, \dots, \quad a_n = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ -1 & \text{if } n \text{ is a multiple of 4,} \\ 1 & \text{if } n \text{ is 2 more than a multiple of 4.} \end{cases}$$

In our carefully structured form, give direct proofs (not just outlines) that:

- (a) S1 is false for A
- (b) S2 is false for A
- (c) S3 is true for A

2. Let  $n$  is a positive integer,  $A$  and  $B$  be Java arrays of length  $n$ .  
Let  $\mathbb{N}$  be the set of nonnegative integers.

- (a) Consider this sentence:

$$S1(A): \forall i \in \mathbb{N}, \forall j \in \mathbb{N}, i \leq j < n \rightarrow A[i] \leq A[j].$$

Write a very short English sentence for  $S1(A)$ .

- (b) Now consider this sentence:

$$S2(A): \exists k \in \mathbb{N}, \left( (k < n) \wedge (\forall i \in \mathbb{N}, \forall j \in \mathbb{N}, i \leq j \leq k \rightarrow A[i] \leq A[j]) \wedge (\forall i \in \mathbb{N}, \forall j \in \mathbb{N}, k \leq i \leq j < n \rightarrow A[i] \geq A[j]) \right).$$

[Aside: Arrays for which  $S2$  is true are called uni-modal.]

Prove or disprove:

- i.  $S1(A)$  implies  $S2(A)$ .
  - ii.  $S2(A)$  implies  $S1(A)$ .
- (c) Now consider this sentence:

$$S3(A,B): \exists k \in \mathbb{N}, \forall i \in \mathbb{N}, i < n \rightarrow A[i] = B[(i + k) \bmod n].$$

[Aside: We say that  $B$  is a rotation of  $A$  exactly when  $S3(A,B)$  is true. The concept of rotating an array should be familiar to students from CSC148/A48 — compare how the `contents` arrays are used in `ArrayQueue` versus `CircularQueue`.]

Prove or disprove:

- i.  $S3(B,A)$  is necessary for  $S3(A,B)$ .
- ii.  $S1(A)$  and  $S3(A,B)$  are sufficient for  $S1(B)$ .
- iii.  $S2(A)$  and  $S3(A,B)$  are sufficient for  $S2(B)$ .

3. The concept of infinity can be tricky to express precisely. One way to say “there are infinitely many things of a certain kind” is to say “for every number  $n$ , there is a thing of that certain kind whose “size” is bigger than  $n$ . For example, to say that “there are infinitely many prime numbers”, we could say, “for every number  $n$ , there is a prime number  $p$  such that  $p > n$ ”. [Aside: You should convince yourself that these two sentences are indeed equivalent.]

Consider the following Java method:

```
public static boolean f3(String s) {
    int state = 0;
    String t = s;
    while (t.length() > 0) {
        if (state == 0) {
            if (t.charAt(0) == 'A') { // if first char is 'A'
                state = 1;
            } else {
                state = 2;
            }
        } else if (state == 1) {
            if (t.charAt(0) == 'B') { // if first char is 'B'
                state = 0;
            } else {
                state = 2;
            }
        }
        t = t.substring(1); // remove first char
    }
    return state == 1;
}
```

- (a) Use precise symbolic notation to express the statement “there are infinitely many strings  $s$  such that  $f3(s)$  returns true”.
- (b) Give a well structured proof for the claim in part (a).

4. The concept of uniqueness can also be tricky to express precisely, at least to the extent that allows us to rigorously prove the uniqueness of something. One way to say “there is a unique thing of a certain kind” (or “there is exactly one thing of a certain kind”) is to say “there is a thing of that certain kind and everything of that kind is that thing”.

Consider the following Java method:

```
public static boolean f4(int n) {
    boolean flag = n>0;
    int t = n;
    int d = 0;
    int i = 0;
    while (i < 9) {
        int dNew = t % 10; // "t mod 10", remainder when t is divided by 10.
        if (dNew <= d)
            flag = false;
        t = t / 10; // integer division, round toward zero.
        d = dNew;
        i = i + 1;
    }
    return t==0 && flag;
}
```

- (a) Use precise symbolic notation to express the statement “there is a unique integer  $n$  such that  $f4(n)$  returns true”.
- (b) Give a well structured proof for the claim in part (a).