Lower Bounds on the Length of Universal Traversal Sequences

(Detailed Abstract)

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Abstract

Universal traversal sequences for d-regular n-vertex graphs require length $\Omega(d^2n^2 + dn^2 \log \frac{n}{d})$, for $3 \leq d \leq n/3 - 2$. This is nearly tight for $d = \Theta(n)$. We also introduce and study several variations on the problem, e.g. edge-universal traversal sequences, showing how improved lower bounds on these would improve the bounds given above.

1. Universal Traversal Sequences

Universal traversal sequences were introduced by Cook (see Aleliunas [1] and Aleliunas *et al.* [2]), motivated by the complexity of graph traversal.

Let $\mathcal{G}(d,n)$ be the set of all connected, d-regular, n-vertex, edge-labeled, undirected graphs $G = (V,E)$. For this definition, edges are labeled as follows. For every edge $\{u,v\} \in E$ there are two labels $l_{u,v}$ and $l_{v,u}$ with the property that, for every $u \in V$, $\{l_{u,v} \mid \{u,v\} \in E\} = \{0,1,\ldots,d-1\}$. For such labeled graphs, a string over $\{0,1,\ldots,d-1\}$ can be thought of as a sequence of edge traversal commands. In particular, any $U = U_1U_2\ldots U_k \in \{0,1,\ldots,d-1\}^*$ and $v_0 \in V$ determine a unique sequence $(v_0,v_1,\ldots,v_k) \in V^{k+1}$ such that $l_{v_{i-1},v_i} = U_i$, for all $i \in \{1,2,\ldots,k\}$. Such a sequence U is said to *traverse* G starting at v_0 if and only if every vertex in G appears at least once in the sequence $v_0, v_1, \ldots, v_k$. Finally, U is a *universal traversal sequence* for $\mathcal{G}(d,n)$ if and only if U traverses each $G \in \mathcal{G}(d,n)$ starting at any vertex in G. $U(d,n)$ denotes the length of the shortest universal traversal sequence for $\mathcal{G}(d,n)$. To avoid certain trivialities, we also define $U(d,n) = U(d,n+1)$ in case $\mathcal{G}(d,n)$ is empty; see Proposition 1.

Universal traversal sequences can also be defined for nonregular graphs of maximum degree d. The restriction to d-regular graphs is largely aesthetic, although the bounds change slightly. See Bar-Noy *et al.* [5] for some results relating the two notions.

There are two connections between universal traversal sequences and the complexity of undirected graph traversal, one motivating construc-

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tive upper bounds and the other motivating lower bounds on $U(d,n)$. Given an undirected graph G and two distinguished vertices s and t, determining whether there is a path from s to t (the problem sometimes known as *USTCON* or *UGAP*) is not known to be solvable in deterministic space $O(\log n)$. The best that is known for this problem is that it can be solved by an errorless probabilistic algorithm running in $O(\log n)$ space and polynomial expected time (Borodin *et al.* [7]). If universal traversal sequences could be constructed in deterministic space $O(\log n)$, then *USTCON* would be solvable within the same bounds. Aleliunas *et al.* [2] demonstrated that polynomial length universal traversal sequences exist, but not by a sufficiently uniform construction. This suffices to demonstrate that *USTCON* can be solved by a nonuniform $O(\log n)$ space algorithm. Uniform constructions of subexponential length are known only for the case $d = 2$ (Istrail [14]) and $d = n - 1$ (Karloff *et al.* [16]); the latter sequences are not of polynomial length. Uniformly constructible sequences of length $n^{o(\log n)}$ would also be very interesting, implying that *USTCON* is solvable in deterministic space $o(\log^2 n)$.

The motivation for studying lower bounds on $U(d,n)$ derives from considering the simultaneous time and space requirements for traversing undirected graphs. It is well known that any graph can be traversed in linear time (but using $\Omega(|V|)$ space) by depth-first search, or $O(\log |V|)$ space (but $\Theta(|V||E|)$ expected time) by random walk (Aleliunas *et al.* [2]). In fact, it has been shown recently that there is a spectrum of time-space compromises between these two endpoints (Broder *et al.* [10]). This raises the intriguing prospect of proving that logarithmic space and linear time are not simultaneously achievable or, more generally, proving a time-space tradeoff that matches these upper bounds.

An appropriate model for proving such a tradeoff would be some variant of the "JAG" of Cook and Rackoff [12]. This is an automaton that has a limited supply of pebbles that it can move from vertex to adjacent vertex. It uses its pebbles to recognize when it has returned to a previously visited vertex. The goal, then, would be to try to prove a tradeoff between the number of pebbles and the number of moves the automaton makes. Of course, an automaton with one pebble could traverse the entire graph merely by moving the pebble according to a universal traversal sequence. Thus, before tackling the problem of time-space tradeoffs we need good lower bounds on the lengths of universal traversal sequences.

We briefly summarize work bounding the length of universal traversal sequences. For convenience, the best known bounds are given in Table 1. Aleliunas *et al.* [2] proved an $O(d^2n^3 \log n)$ upper bound. This bound actually applies to both regular and nonregular graphs. Kahn *et al.* [15] improved this by a factor of d for regular graphs. For the special case $d = 2$ (the cycle), Aleliunas [1] showed an $O(n^3)$ bound. Bar-Noy *et al.* [5] and Bridgland [9] provided constructive, but nonpolynomial, upper bounds for the cycle, recently subsumed by the $O(n^{4.76})$ construction of Istrail [14]. For the special case $d = n-1$ (the clique), Bar-Noy *et al.* [5] showed an $O(n^3 \log^2 n)$ bound, subsequently improved by Alon and Ravid [3] by a factor of $\log n$. Chandra *et al.* [11] have recently shown that the latter bound holds for all $d \geq n/2$. The best constructive bound for the clique is the $n^{O(\log n)}$ bound of Karloff *et al.* [16].

There has been less progress on lower bounds. Bar-Noy *et al.* [5] proved an $\Omega(n \log^2 n / \log\log n)$ bound for the clique. Alon and Ravid [3] improved this to $\Omega(n^2/\log n)$, which holds for all $d = \Omega(n)$. Prior to the current work, the best lower bound for $2 \leq d \leq n/2 - 1$ was the following, also proved by Bar-Noy *et al.* [5]:

$$U(d,n) = \Omega(dn + n \log n). \tag{1}$$

This is still the best bound for $d = 2$, but for $3 \leq d \leq n/3 - 2$ we improve this lower bound to

$$U(d,n) = \Omega(d^2n^2 + dn^2 \log \frac{n}{d}).$$

In particular, for constant degree the lower bound is improved from $\Omega(n \log n)$ to $\Omega(n^2 \log n)$, and for linear degree $d \leq n/3 - 2$ from $\Omega(n^2)$ to $\Omega(n^4)$. Note that the latter differs from the upper bound $O(n^4 \log n)$ only by a logarithmic factor. We also give cubic lower bounds which hold for infinitely many pairs d,n with $n/3 - 2 < d \leq n/2 - 1$.

One important technical point to be considered is that d-regular, n-vertex graphs do not exist for all values of d and n.

Bound	Range of Validity	Source
$U(d,n) = O(n^3)$	$d = 2$	Aleliunas [1]
$U(d,n) = O(dn^3 \log n)$	$3 \leq d \leq n/2 - 1$	Kahn *et al.* [15]
$U(d,n) = O(n^3 \log n)$	$n/2 \leq d$	Chandra *et al.* [11]
$U(d,n) = \Omega(n \log n)$	$d = 2$	Bar-Noy *et al.* [5]
$U(d,n) = \Omega(d^2n^2 + dn^2 \log \frac{n}{d})$	$3 \leq d \leq n/3 - 2$	This paper
$U(d,n) \geq \text{const} \cdot n^3$ i.o.	$n/3 - 2 < d \leq n/2 - 1$	This paper
$U(d,n) = \Omega(n^2)$	$n/3 - 2 < d \leq n/2 - 1$	Bar-Noy *et al.* [5]
$U(d,n) = \Omega(n^2/\log n)$	$n/2 - 1 < d$	Alon and Ravid [3]

Table 1: Bounds on Length of Universal Traversal Sequences

Proposition 1: For all d and n, d-regular n-vertex graphs exist if and only if $0 \leq d < n$ and dn is even.

Proof: (See, for example, Lovász, [17, exercise 5.2].) For the "only if" clause, $dn/2$ is the number of edges, which must be an integer. For the "if" clause, let $V = \{0, 1, \ldots, n-1\}$ and $E = \{\{i, (i+j) \bmod n\} \mid 0 \leq i < n \text{ and } 1 \leq j \leq \lfloor d/2 \rfloor\}$. If d is even, (V, E) is d-regular. If d is odd, then n must be even, so replace E by $E \cup \{\{i, (i + n/2) \bmod n\} \mid 0 \leq i < n\}$. □

In order to talk about Ω bounds on $U(d,n)$, define $U(d,n) = U(d, n+1)$ whenever dn is odd.

The remainder of the paper is organized as follows. In Section 2 we give our basic lower bound argument, proving the $\Omega(d^2n^2)$ bound. Section 3 proves a technical result, showing that $U(d,n)$ is nearly monotone in n. This is needed in several of our results to transform infinitely-often lower bounds into almost-everywhere (Ω) lower bounds. Section 4 proves the $\Omega(dn^2 \log \frac{n}{d})$ term of our lower bound, by reducing to the problem (defined there) of "circumnavigating" a cycle many times, and generalizing the cycle lower bound of Bar-Noy *et al.* [5]. Section 5 further generalizes the "circumnavigation" reduction, giving possible approaches to improving our lower bounds. Section 6 concludes with open problems.

2. The $\Omega(d^2n^2)$ Lower Bound

In this section we present the basic lower bound argument for universal traversal sequences. It is used to prove the following two theorems. Where applicable, Theorem 2 is generally the stronger result, but Theorem 3 extends over a wider range of degrees (and provides slightly better absolute bounds for certain small values of n and d).

Theorem 2: For all $3 \leq d \leq n/3 - 2$, $U(d,n) = \Omega(d^2n^2)$. In particular, let dn be even, and let d' be the least integer in the range $d + 1 \leq d' \leq d + 4$ such that $n - d'$ and $d(n - d')/2$ are even. (Such a d' exists, since it suffices for $n - d'$ to be a multiple of 4.) If $3 \leq d \leq (n - 2 - (d' - d))/3$, then

$$U(d,n) \geq \frac{d(d-2)(n-d')^2 + 4d(n-d')}{16}. \qquad (2)$$

Theorem 3: For all $3 \leq d \leq n/3 - 1$, $U(d,n) = \Omega(dn^2)$. In particular, for $3 \leq d \leq n/2 - 1$, n even, and $dn/2$ even,

$$U(d,n) \geq \frac{(d-2)n^2 + 4n}{8}. \qquad (3)$$

We will concentrate on the proof of Theorem 2; the proof of Theorem 3 is very similar. The following definitions will be useful.

Definition: A sequence U is *edge-universal* for $\mathcal{G}(d,n)$ if, from all starting vertices of all graphs G in $\mathcal{G}(d,n)$, the path defined by U includes each (undirected) edge of G at least once. U is *s-edge-universal (s-vertex-universal)* if the path defined by U includes each edge (respectively, enters each vertex) at least s times.

The first observation is that if a sequence U is universal, then it must also be edge-universal for slightly smaller graphs. This is implicit in the proof of Lemma 13 of Bar-Noy *et al.* [5]. (Their lemma supplies the $\Omega(dn)$ bound of Equation 1 in Section 1.)

Lemma 4 (Bar-Noy *et al.*): For dn even, if U is universal for $\mathcal{G}(d,n)$, then U is edge-universal for $\mathcal{G}(d,n-d')$ for all $d' \geq d+1$.

Proof: If $\mathcal{G}(d,n-d')$ is empty, then the conclusion holds vacuously. Otherwise, we proceed by contradiction. Let (H, v_0, e) be a counterexample, i.e., $H \in \mathcal{G}(d,n-d')$ is a graph with a vertex v_0 and an edge e such that U starting from v_0 fails to traverse edge e. By "hiding" some vertices on edge e, we can build from H a graph G in $\mathcal{G}(d,n)$ for which U fails to be universal. Placing one vertex in the "middle" of e would suffice, except that the graph would no longer be d-regular. Instead, we attach an arbitrarily labeled d-regular, d'-vertex graph K. (By assumption $d < d'$, and if d is odd, then n and $n-d'$ must be even, whence d' is even, so such a K exists by Proposition 1.) Join H and K by removing $e = \{u, v\}$ and any edge $\{y, z\}$ of K, and adding the edges $\{u, y\}$ and $\{v, z\}$ so that the resulting graph G is connected. Now U starting at v_0 in G will behave exactly as in H, never leaving either of the vertices incident to e by the label which would have crossed e, and thus will never enter K. This contradicts the universality of U.

Note that a $(d+1)$-clique is the smallest d-regular graph K, so d' must at least $d+1$. $\square$

The key idea in the lower bound technique is found in the following lemma, which shows that an edge-universal sequence must be "highly" edge universal for smaller graphs.

Lemma 5: Let n be even. If U is edge-universal for $\mathcal{G}(d,n)$, then it is s-edge-universal for $\mathcal{G}(d,n/2)$, where $s = (d-2)n/4 + 1$.

Proof: The theorem is vacuously true if $\mathcal{G}(d,n/2)$ is empty. Otherwise, the proof is by contradiction. Let (H, v_0, e) be a counterexample, i.e., $H = (V_H, E_H)$ is a graph in $\mathcal{G}(d,n/2)$ with a vertex v_0 and an edge e such that U starting from v_0 crosses e only t times, where $t < s$.

Partition the edges of H into two sets C and S so that (V_H, C) is connected and contains e. In particular, let C be any spanning tree of H containing e. The edges in S will be called "switchable" edges, for reasons to be made clear below. Note that $|S| = dn/4 - (n/2 - 1) = s$.

Define a family $\{G_x \mid x \in \{0,1\}^s\} \subseteq \mathcal{G}(d,n)$ as follows. $G_{\{0\}^s}$ is simply the graph consisting of two disjoint copies of H. For each $x \neq \{0\}^s$, G_x is similar except that certain pairs of switchable edges, one from each copy of H, are crossed from one copy to the other. (See Figure 1.) These pairs of edges are selected from S as dictated by 1's in the corresponding positions of x. (A special case of this construction appears in a different context in Awerbuch *et al.* [4].)

More precisely, $G_x = (V, E_x)$ is defined as follows.

Choose a one-to-one correspondence between edges in S and bit positions in x. For $u, v \in V_H$ let $x_{u,v}$ be the bit corresponding to edge $\{u, v\}$ if $\{u, v\} \in S$; otherwise $x_{u,v} = 0$. Let $\oplus$ denote the EXCLUSIVE OR operation. Let

$$V^i = \{v^i \mid v \in V_H\}, \quad i \in \{0,1\}.$$

Then finally we have

$$\begin{aligned} V &= V^0 \cup V^1 \\ E_x &= \{\{u^i, v^{i \oplus x_{u,v}}\} \mid \{u,v\} \in E_H, \\ &\qquad \text{and } i \in \{0,1\}\}, \end{aligned}$$

and, for all $\{u, v\} \in E_H$ and $i \in \{0,1\}$,

$$l_{u^i, v^{i \oplus x_{u,v}}} = l_{u,v}.$$

The vertices in V^0 will be referred to as the "left hand" copy of H, and those in V^1 as the "right hand" copy. Note that G_x is connected for all $x \neq \{0\}^s$, since each copy of H is internally connected via the unswitchable spanning tree edges, and the two copies are connected to each other through at least one switched edge.

The key observation about this family of graphs is that for any sequence U, the path followed by U in H is identical to the path followed by U in G_x, except that in G_x the path will cross between the left and right copies of H on some steps. This is easy to see from the definition of E_x: if the path leaves vertex u along edge $\{u, v\}$ in H at some step,

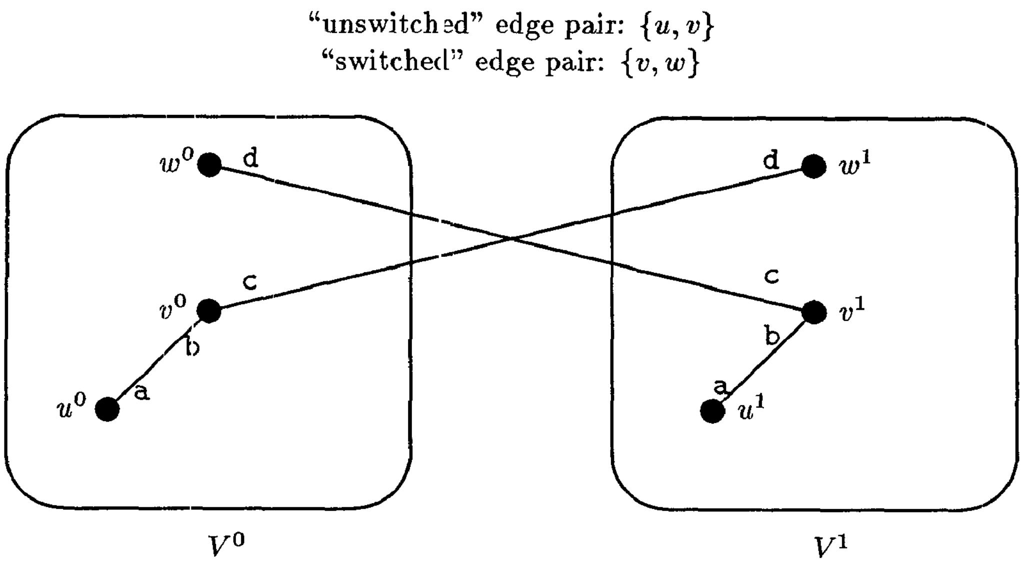

Figure 1: G_x and Switchable Edges

then no matter whether it's at u^0 or u^1 in G_x at the same step, and no matter whether $x_{u,v}$ is 0 or 1, it will be at either v^0 or v^1 at the end of the step.

In fact, we can say more. Define the *parity* of an edge to be 0 if both end points are in the same copy of H, and 1 if the two end points are in different copies. In other words, the parity of the edge $\{u,v\}$ is $x_{u,v}$. Suppose the sequence of vertices visited by U in H starting at v_0 is

$$v_0, v_1, v_2, \ldots.$$

Then in G_x starting at v_0^0, U will visit the sequence

$$v_0^0, v_1^{p_1}, v_2^{p_2}, \ldots$$

where $p_j \in \{0,1\}$ is the net parity of all the edge crossings up to and including the j^{th} step, i. e.,

$$p_j = x_{v_0,v_1} \oplus x_{v_1,v_2} \oplus \cdots \oplus x_{v_{j-1},v_j}. \qquad (4)$$

This fact is easily proved by induction on j.

We are now prepared to show the central claim: if U when started from v_0 in H traverses $e = \{u,v\}$ a number $t < s$ of times, then there is an $x \neq \{0\}^s$ such that U in G_x started from the left hand copy of v_0, namely v_0^0, *never* traverses the right hand copy of e, namely $\{u^1, v^1\}$. (Note that e is a non-switchable edge, by construction, so its two copies in G_x don't cross between copies of H.) Suppose e is traversed during steps $j_1, \ldots, j_t$ and no others. Thus

$$\begin{aligned}\{u,v\} = \{v_{j_1-1}, v_{j_1}\} &= \{v_{j_2-1}, v_{j_2}\} \\ = \quad \cdots \quad &= \{v_{j_t-1}, v_{j_t}\}\end{aligned}$$

and this is true of no other pair $\{v_{j-1}, v_j\}$. Then choose an $x \neq \{0\}^s$ such that

$$\begin{aligned} p_{j_1} &= 0 \\ p_{j_2} &= 0 \\ &\vdots \\ p_{j_t} &= 0. \end{aligned}$$

From Equation 4 this is a system of t homogeneous linear equations in s unknowns over $GF(2)$. Since $t < s$, this system always has a nonzero solution (Herstein [13, Corollary to Theorem 4.3.3]). □

We can now prove Theorem 2.

Proof of Theorem 2: As in the statement of the theorem, let dn be even and d' be the least integer satisfying $d+1 \leq d' \leq d+4$ such that both $n-d'$ and $d(n-d')/2$ are even. If U is universal for $\mathcal{G}(d,n)$, then by Lemma 4 it is edge-universal for $\mathcal{G}(d, n-d')$, and so by Lemma 5 it is s-edge-universal for $\mathcal{G}(d,(n-d')/2)$, where $s = (d-2)(n-d')/4 + 1$. Clearly, an s-edge-universal sequence for $\mathcal{G}(d,(n-d')/2)$ must have length at least s times the number of edges in graphs in

$\mathcal{G}(d,(n-d')/2)$, i. e.,

$$\begin{aligned}|U| &\geq sd(n-d')/4\\ &= \frac{d(d-2)(n-d')^2+4d(n-d')}{16}.\end{aligned}$$

It is straightforward to verify that $\mathcal{G}(d,n)$, $\mathcal{G}(d,n-d')$, and $\mathcal{G}(d,(n-d')/2)$ are all nonempty, due to the various evenness constraints, and the assumption that $d \leq (n-2-(d'-d))/3$, which is equivalent to $d \leq (n-d')/2-1$.

The stated Ω bound follows since $d' \leq d+4$ and $n/3-2 \leq (n-2-(d'-d))/3$. □

The proof above is not valid for $d > n/3-1$, since Lemma 4 requires the insertion of a large gadget when d is large. However, the technique of Lemma 5 can be applied to obtain the lower bound of Theorem 3 for degrees up to $n/2-1$, by hiding a vertex rather than an edge.

Lemma 6: Let n be even. If U is universal for $\mathcal{G}(d,n)$, then U is s-vertex-universal for $\mathcal{G}(d,n/2)$, where $s=(d-2)n/4+1$.

Proof (Sketch): The proof is essentially the same as the proof of Lemma 5, except that rather than choosing an infrequently traversed edge to avoid in the right-hand copy of H in G_x, one chooses an infrequently visited vertex. □

The proof of Equation (3) of Theorem 3 is then immediate: if U is s-vertex-universal for $n/2$ vertex graphs, then $|U| \geq sn/2$.

Perhaps somewhat surprisingly, $U(d,n)$ is not known to be monotone in n. Thus, the lower bound for infinitely many values of d and n given above does not immediately imply the Ω lower bound (i. e., for almost all n) stated in Theorem 3. However, we can show that $U(d,n)$ is "sufficiently monotone" to yield the stated Ω bound, for d up to $n/3-1$. This is deferred to Section 3.

Even in the range $n/3-1 < d \leq n/2-1$ where the almost everywhere bound does not hold, Theorem 3 still provides a "dense" lower bound, valid for half of the d,n pairs having dn even: namely, those with $dn \equiv 0 \pmod 4$ and n even.

The bounds given in Equations 2 and 3 are valid for $d=2$, but trivial. The underlying reason is that Lemma 5's spanning tree would then contain all but one edge of H. Making more edges switchable could easily leave the graph disconnected.

3. $U(d,n)$ is Nearly Monotone in n

Intuitively, one would expect that $U(d,n)$ is monotonically nondecreasing with n, but there is currently no proof of this conjecture, except for the easy case of $d=2$ (e.g., see Aleliunas [1] or Theorem 7 below), and the case $d=3$, which follows from Theorem 7 below. In the full paper [8] we prove Theorem 7, which shows that $U(d,n)$ is "monotone in the large", although there is still the possibility that it is nonmonotone within small regions.

The idea for our proof of Theorem 7 came from a construction due to Steve Mann [personal communication].

Theorem 7: For all d,n, and all $b \geq d-1$, $U(d,n) \leq U(d,n+b)$.

In addition to its intrinsic interest, this weak monotonicity result can be used to parlay "infinitely often" lower bounds, such as the ones presented in Theorem 3 and in Section 4, into "almost everywhere" (Ω) lower bounds.

4. The $\Omega(dn^2 \log\frac{n}{d})$ Lower Bound

The $\Omega(dn^2\log\frac{n}{d})$ lower bound begins with many of the same ideas used in Section 2. By a careful choice of the graph H and its s switchable edges in Lemma 5, Section 4 shows that, from any universal traversal sequence of length u for the family $\{G_x \mid x \in \{0,1\}^s\}$, we can extract a sequence over $\{0,1\}$ of length $O(u/d)$ that "circumnavigates" any labeled $\frac{n}{8(d-1)}$-cycle $\Omega(dn)$ times. Bar-Noy *et al.* [5] prove that one circumnavigation of such a cycle requires a sequence of length $\Omega(\frac{n}{d}\log\frac{n}{d})$. Section 4.2 generalizes their lemmas to prove that t circumnavigations requires a sequence whose length is t times as great. Hence, $u/d = \Omega(n^2\log\frac{n}{d})$.

Given the amount of technical detail required to rework the lemmas of Bar-Noy *et al.* [5], the resulting gain over the bound of Section 2 may appear small. However, the reduction from universal sequences to multiple circumnavigations is very general, and may well lead to dramatically improved lower bounds. (See Section 6.)

4.1. Reduction to Circumnavigations

For any labeled cycle $C \in \mathcal{G}(2,n)$, a string over $\{0,1\}$ can be interpreted as a traversal sequence. In particular, any $U \in \{0,1\}^*$ and start vertex v_0 of C determine a unique sequence $(v_0, v_1, \ldots, v_k)$ of vertices traversed by U. Such a sequence U is said to circumnavigate C t times starting at v_0 if there are at least t times at which the sequence returns to v_0 moving in the same direction in which it last left v_0. More precisely, U *circumnavigates* C t *times* if and only if there exist $0 \leq i_1 < i_2 < \cdots < i_{2t} \leq k$ such that

1. $v_0 = v_{i_1} = v_{i_2} = \cdots = v_{i_{2t}}$,
2. $v_l \neq v_0$ for all $i_{2j-1} < l < i_{2j}$ and $1 \leq j \leq t$, and
3. $v_{i_{2j-1}+1} \neq v_{i_{2j}-1}$, for all $1 \leq j \leq t$.

U is a *t-circumnavigation sequence* for $\mathcal{G}(2,n)$ if and only if U circumnavigates each $C \in \mathcal{G}(2,n)$ t times starting at any vertex in C. $C(t,n)$ denotes the length of the shortest t-circumnavigation sequence for $\mathcal{G}(2,n)$.

Theorem 8: Let n be a multiple of $8(d-1)$. Then

$$U(d,n) \geq \frac{d}{2} C(2s, m),$$

where $s = \frac{(d-2)n}{8}$ and $m = \frac{n}{8(d-1)}$.

Proof: This proof combines ideas from the proofs of Lemma 6 and Bar-Noy *et al.* [5, Lemma 9].

For any $\alpha \in \{0, 1, \ldots, d-1\}^*$, let $\alpha|_{0,1}$ be the result of deleting all symbols other than 0 and 1 from α. Let U be a universal traversal sequence for $\mathcal{G}(d,n)$. Assume without loss of generality that 0 and 1 are the two least frequently occurring symbols in U, so that $|U| \geq \frac{d}{2}|U|_{0,1}|$. Let $C \in \mathcal{G}(2,m)$ be an arbitrary labeled cycle, and v an arbitrary starting vertex of C. It suffices, then, to prove that $U|_{0,1}$ circumnavigates C $2s$ times starting at v.

Construct $H \in \mathcal{G}(d, n/2)$ as follows. Let $K^i_{d-1} = (V^i, E^i)$ for $0 \leq i < 4m$ be disjoint copies of the $(d-1)$-clique K_{d-1}. Let $V^i = \{v^i_1, v^i_2, \ldots, v^i_{d-1}\}$. Then

$$H = \left(\bigcup_{i=0}^{4m-1} V^i, \left(\bigcup_{i=0}^{4m-1} E^i \right) \cup \left(\bigcup_{i=0}^{4m-1} D^i \right) \right),$$

where

$$D^i = \{\{v^i_j, v_j^{(i+1) \bmod 4m}\} \mid 1 \leq j \leq d-1\}.$$

Label the edges in $\cup_i E^i$ arbitrarily from $\{2, 3, \ldots, d-1\}$. If β is the string of length m that circumnavigates C starting at v once in a clockwise direction, then label the edges in $\cup_i D^i$ so that $\beta\beta\beta\beta$ circumnavigates the cycle $(v^0_j, v^1_j, \ldots, v^{4m-1}_j)$ starting at v^0_j once in a clockwise direction, for all $1 \leq j \leq d-1$. There is an obvious homomorphism ϕ from H to C that maps four cliques into each vertex of C, and such that if a sequence α starting at v^0_1 ends in some clique K^i_{d-1}, then $\alpha|_{0,1}$ starting at v ends at the image under ϕ of K^i_{d-1}.

Let T be any spanning tree of H, and let the set of switchable edges of H be

$$S = \left(\bigcup_{i=0}^{2m} E^i \right) \cup \left(\bigcup_{i=0}^{2m-1} D^i \right) - T,$$

that is, almost all edges in half of H. Note that

$$\begin{aligned} |S| &\geq \frac{dn}{8} + \frac{(d-1)(d-2)}{2} - \left(\frac{n}{4} + d - 2\right) \\ &\geq (d-2)n/8 = s. \end{aligned}$$

For any $x \in \{0,1\}^s - \{0\}^s$, construct $G_x \in \mathcal{G}(d,n)$ from two disjoint copies of H as in Lemma 5. (If $|S| > s$, then by convention the switchable edges beyond the first s are never switched.)

Consider the vertex v^{3m}_1, which is in the clique farthest from any switchable edge. It must be the case that U, when applied to H starting at v^0_1, makes at least s traversals from S to v^{3m}_1 and back to S. If not, as in Lemma 5, there is an $x \neq \{0\}^s$ such that U, when applied to G_x starting at the left hand copy of v^0_1, never reaches the right hand copy of v^{3m}_1, contradicting the universality of U for $\mathcal{G}(d,n)$. Since the homomorphism ϕ maps K^0_{d-1}, K^{2m}_{d-1}, and v^{3m}_1 onto v, these s traversals back and forth are mapped into $2s$ circumnavigations of C starting at v. □

4.2. A Lower Bound on t-Circumnavigating an n-Cycle

In this section, we generalize the cycle lower bound of Bar-Noy *et al.* [5] to circumnavigations. Our presentation is self contained, but closely follows their proof.

Definition: A *labeling* $w \in \{0,1\}^*$ is a labeled chain $(0, 1, \ldots, |w|)$ of vertices such that $l_{i-1,i} = w_i$, for $1 \leq i \leq |w|$.

A labeling might, for example, represent an arc of a labeled cycle.

Definition: A traversal sequence $\alpha \in \{0,1\}^*$ *traverses* a labeling w if and only if, when started at the left end of w, it reaches the right end of w (without falling off the left end) exactly at the end of α. To make this more precise, suppose that α, beginning at vertex 0 of the labeling, visits the sequence $(v_0 = 0, v_1, \ldots, v_{|\alpha|})$ of vertices. Then, for all $1 \leq j \leq |\alpha|$,

1. $0 \leq v_j \leq |w|$,
2. $\alpha_j = l_{v_{j-1},v_j}$, and
3. $v_j = |w|$ if and only if $j = |\alpha|$.

Lemma 9: If α traverses u and β traverses v, then $\alpha\beta$ traverses uv. Conversely, if γ traverses uv, then $\gamma = \alpha\nu\beta$, where α traverses u, and β traverses v.

Proof: The forward direction is immediate. For the converse, let α (β) be the prefix (suffix) of γ up to (after) the first entry into (last departure from) the vertex at the boundary between u and v. □

Definition: A sequence β is an *a-block* if β traverses 0^a1^a, but no proper suffix of β does so. The minimal prefix (suffix) of β traversing 0^a (1^a) is called an *a-half-block*, and is denoted β^0 (β^1, respectively).

Definition: For $\beta \in \{0,1\}^*$ and $x \in \{0,1\}$, let $\#_x\beta$ be the number of occurrences of x in β.

Lemma 10 (See [5, Lemma 3]): Let β be an a-block. Then

(10.1) $\#_0\beta^0 - \#_1\beta^0 = a$, and $\#_1\beta^1 - \#_0\beta^1 = a$, and

(10.2) every nonempty prefix and suffix of $\beta^0(\beta^1)$ has more 0's (1's) than 1's (0's).

Proof: Condition 10.1 is necessary in order to traverse 0^a and 1^a. Condition 10.2 follows from the minimality of blocks and half-blocks, and from the requirement that they not "fall off the ends" of the labeling being traversed. □

Following the notation in [5], we identify a block or half block with its set of (consecutive) indices in the sequence α. For example, if β and γ are blocks in α, we use the set notation $\beta \subseteq \gamma$ to denote that β is a subinterval of γ.

Lemma 11 (See [5, Lemma 4]): For any $w_0, w_1, \ldots, w_m$, if α traverses $w_0 \prod_{i=1}^{m}(0^a1^aw_i)$, then α contains m pairwise disjoint a-blocks. ($\prod$ denotes string concatenation.)

Proof: Let $\alpha = \alpha_0\alpha_1 \cdots \alpha_m$, where $\alpha_0\alpha_1 \cdots \alpha_i$ is the prefix of α up to and including the symbol entering the last vertex in $w_0 \prod_{j=1}^{i}(0^a1^aw_j)$ for the last time. Then α_{i+1} starts with an a-block. □

Lemma 12: Let U be a t-circumnavigation sequence for $\mathcal{G}(2,n)$. For every $1 \leq a \leq n/4$, let $m_a = \left(\lfloor \frac{n}{2a} \rfloor - 1\right)t$. Then there exist strings $w_{a,0}, w_{a,1}, \ldots, w_{a,m_a}$ such that U traverses each labeling in the set $\{w_a \mid 1 \leq a \leq n/4\}$, where

$$w_a = w_{a,0} \prod_{i=1}^{m_a} (0^a1^aw_{a,i}).$$

Proof: Let $C_a \in \mathcal{G}(2,n)$ be the cycle labeled (clockwise, from a designated start vertex v_0) by the n-symbol prefix of $(0^a1^a)^{\lceil \frac{n}{2a} \rceil}$. Let c_a denote the clockwise labeling of C_a starting from v_0, and $\overline{c_a}$ denote the counterclockwise labeling. Thus, if $c_a = (0^a1^a)^kx$, where $|x| < 2a$, then $\overline{c_a} = 1y(0^a1^a)^{k-1}(0^a1^{a-1})$, where y is the complement of the reverse of x.

Let $U = \epsilon_1\gamma_1\epsilon_2\gamma_2 \cdots \epsilon_t\gamma_t\epsilon_{t+1}$ where, for $1 \leq j \leq t$, ϵ_j (possible empty) traverses C_a from v_0 back to v_0 zero or more times without completing a circumnavigation, whereas γ_j completes exactly one circumnavigation, starting and ending at v_0, and not visiting v_0 otherwise. Then γ_j traverses c_a (if γ_j was a clockwise circumnavigation) or $\overline{c_a}$ (if γ_j was a counterclockwise circumnavigation). As noted above, c_a contains $(0^a1^a)^{\lfloor n/2a \rfloor}$, and $\overline{c_a}$ contains $(0^a1^a)^{\lfloor n/2a \rfloor - 1}$. In either case, there exist y_j and z_j such that γ_j traverses $y_j(0^a1^a)^{\lfloor n/2a \rfloor - 1}z_j$. Obviously, ϵ_j traverses ϵ_j. Thus, by Lemma 9, $U = \left(\prod_{j=1}^{t} \epsilon_j\gamma_j\right)\epsilon_{t+1}$ traverses $\left(\prod_{j=1}^{t} \epsilon_jy_j(0^a1^a)^{\lfloor n/2a \rfloor - 1}z_j\right)\epsilon_{t+1}$. The lemma follows by collecting the ϵ_j's, y_j's, and z_j's into the appropriate $w_{a,i}$'s. □

Definition: Two half blocks have a *trivial intersection* if and only if they are either disjoint or one is contained in the other.

Lemma 13 (See [5, Lemma 5]): Let β and $\tilde{\beta}$ be two blocks. Then β^0 and $\tilde{\beta}^1$ have a trivial intersection.

Proof: This follows from condition 10.2 of Lemma 10, the prefix and suffix properties of β^0 and $\tilde{\beta}^1$. $\square$

Definition: A set $\{\beta_j^{x_j} \mid 1 \leq j \leq r \text{ and } x_j \in \{0,1\}\}$ of half blocks is *nested* if and only if

1. every pair of half blocks has a trivial intersection, and
2. if $\beta_j^x \subseteq \beta_k^x$ for $j \neq k$, then there exists an $l \notin \{j,k\}$ such that $\beta_j^x \subseteq \beta_l^{\overline{x}} \subseteq \beta_k^x$, where $\overline{x} = 1 - x$.

Lemma 14 (See [5, Lemma 6]): For $1 \leq a \leq n/4$, let $m_a = (\lfloor \frac{n}{2a} \rfloor - 1)\,t$, let $w_{a,0}, w_{a,1}, \ldots, w_{a,m_a}$ be strings in $\{0,1\}^*$, and let $w_a = w_{a,0}\prod_{i=1}^{m_a}(0^a 1^a w_{a,i})$. If U traverses each labeling in the set $\{w_a \mid 1 \leq a \leq n/4\}$ then U contains a nested set of half blocks

$$\{\beta_{ik}^{x_{ik}} \mid 1 \leq i \leq n/4,\ 1 \leq k \leq t\},$$

where, letting $a_i = \lfloor n/4i \rfloor$, $\beta_{ik}^{x_{ik}}$ traverses $(x_{ik})^{a_i}$.

Proof: By induction on i.

Let B_{a_i} be a maximum cardinality set of pairwise disjoint a_i-blocks in U. By Lemma 11 and the fact that U traverses w_{a_i},

$$\begin{aligned}
|B_{a_i}| &\geq m_{a_i} \\
&= \left(\left\lfloor \frac{n}{2\lfloor n/4i \rfloor} \right\rfloor - 1\right)t \\
&\geq \left(\left\lfloor \frac{n}{2(n/4i)} \right\rfloor - 1\right)t \\
&= (2i-1)t \\
&\geq it
\end{aligned}$$

for $i \geq 1$. When $i = 1$, pick any half block of each of t a_1-blocks in B_{a_1}. Assume the lemma is true for $i-1$, and let

$$B = \{\beta_{jk}^{x_{jk}} \mid 1 \leq j \leq i-1 \text{ and } 1 \leq k \leq t\}$$

be the nested set asserted by the induction hypothesis. We will show how to find t half blocks of B_{a_i} that preserve the nestedness of B.

For each $\beta \in B_{a_i}$ define

$$In(\beta) = \{\beta_{jk}^{x_{jk}} \in B \mid \beta^{x_{jk}} \cap \beta_{jk}^{x_{jk}} \neq \emptyset \text{ and } \beta^{\overline{x_{jk}}} \not\subseteq \beta_{jk}^{x_{jk}}\}$$

As will be seen, $In(\beta)$ includes all the half blocks that could possibly "interfere with" β, i.e. prevent either half block of β from being included in B.

Claim 1: The sets $In(\beta)$, $\beta \in B_{a_i}$, are pairwise disjoint. To see this, suppose by way of contradiction that $\beta_{jk}^{x_{jk}} \in In(\beta) \cap In(\tilde{\beta})$ for some $\beta \neq \tilde{\beta}$. Without loss of generality, assume that β occurs in U to the left of $\tilde{\beta}$. $\beta_{jk}^{x_{jk}}$ has nonempty intersection with $\beta^{x_{jk}}$ and $\tilde{\beta}^{x_{jk}}$ but contains neither $\beta^{\overline{x_{jk}}}$ nor $\tilde{\beta}^{\overline{x_{jk}}}$, which is impossible, since one of $\beta^{\overline{x_{jk}}}$ and $\tilde{\beta}^{\overline{x_{jk}}}$ lies between $\beta^{x_{jk}}$ and $\tilde{\beta}^{x_{jk}}$.

Claim 2: There exist t blocks $\beta \in B_{a_i}$ such that $In(\beta) = \emptyset$. This is true by Claim 1 and the facts that there are at least it a_i-blocks in B_{a_i} and exactly $(i-1)t$ half blocks in B.

For $1 \leq l \leq t$, let β_{il} be an a_i-block in B_{a_i} such that $In(\beta_{il}) = \emptyset$. If $\beta_{il}^0 \cup \beta_{il}^1$ is disjoint from every $\beta_{jk}^{x_{jk}} \in B$, we can pick either half block of β_{il}, so we arbitrarily let $x_{il} = 0$. Otherwise, consider a minimal (in the inclusion sense) $\beta_{jk}^{x_{jk}} \in B$ such that $\beta_{jk}^{x_{jk}} \cap (\beta_{il}^0 \cup \beta_{il}^1) \neq \emptyset$. Then since $\beta_{jk}^{x_{jk}} \notin In(\beta_{il})$ and by Lemmas 10 and 13, we must have $\beta_{il}^{\overline{x_{jk}}} \subseteq \beta_{jk}^{x_{jk}}$, so let $x_{il} = \overline{x_{jk}}$. The fact that

$$\{\beta_{jk}^{x_{jk}} \mid 1 \leq j \leq i-1 \,\&\, 1 \leq k \leq t\} \cup \{\beta_{il}^{x_{il}} \mid 1 \leq l \leq t\}$$

is properly nested follows from the induction hypothesis and the pairwise disjointness of $\beta_{il}^{x_{il}}$, $1 \leq l \leq t$. $\square$

Lemma 15 (See [5, Lemma 7]): Let $B = \{\beta_j^{x_j} \mid 1 \leq j \leq r\}$ be a nested set of half-blocks, and for $1 \leq j \leq r$, let b_j be such that $\beta_j^{x_j}$ traverses $(x_j)^{b_j}$. Then

$$\left|\bigcup_{j=1}^{r} \beta_j^{x_j}\right| \geq \sum_{j=1}^{r} b_j.$$

Proof: Without loss of generality, assume that the half blocks in B are numbered so that $\beta_i^{x_i}$ is not contained in $\beta_j^{x_j}$ for $j < i$. The proof is by induction on r. The case $r = 1$ follows immediately from condition 10.1 of Lemma 10. Assume the lemma is true for $r-1$, so that $|\bigcup_{j=1}^{r-1} \beta_j^{x_j}| \geq \sum_{j=1}^{r-1} b_j$.

By part 2 in the definition of nested sequences, the maximal $\beta_j^{x_j} \subseteq \beta_r^{x_r}$ are of opposite type, i. e. $x_j = \overline{x_r}$. Let β be the union of the $\beta_j^{\overline{x_r}}$ that are maximal half blocks contained in $\beta_r^{x_r}$. In order for $\beta_r^{x_r}$ to satisfy condition 10.1 of Lemma 10, it must be the case that $|\beta_r^{x_r} - \beta| \geq b_r$, so that

$$\begin{aligned} |\bigcup_{j=1}^{r} \beta_j^{x_j}| &= |(\bigcup_{j=1}^{r-1} \beta_j^{x_j}) \cup \beta_r^{x_r}| \\ &= |(\bigcup_{j=1}^{r-1} \beta_j^{x_j}) \cup (\beta_r^{x_r} - \beta)| \\ &= |\bigcup_{j=1}^{r-1} \beta_j^{x_j}| + |\beta_r^{x_r} - \beta| \\ &\geq \sum_{j=1}^{r-1} b_j + b_r = \sum_{j=1}^{r} b_j. \end{aligned}$$

□

Theorem 16 (See [5, Theorem 2]): If U is a t-circumnavigation sequence for $\mathcal{G}(2,n)$, then $|U| \geq \frac{1}{4}tn(\ln n - O(1))$. That is, $C(t,n) = \Omega(tn \log n)$.

Proof: From Lemmas 12 and 14 it is immediate that U contains a nested set of half blocks that includes t distinct a_i-half blocks for each $1 \leq i \leq n/4$. Thus, from Lemma 15,

$$\begin{aligned} |U| &\geq t\sum_{i=1}^{n/4} a_i = t\sum_{i=1}^{n/4} \left\lfloor \frac{n}{4i} \right\rfloor \\ &\geq \frac{1}{4}tn\left(\left(\sum_{i=1}^{n/4} \frac{1}{i}\right) - 1\right) \\ &\geq \frac{1}{4}tn(\ln n - O(1)). \end{aligned}$$

□

Corollary 17: If $3 \leq d = o(n)$, then $U(d,n) = \Omega(dn^2 \log \frac{n}{d})$.

Proof: From Theorems 8 and 16, whenever $8(d-1)$ divides n,

$$U(d,n) \geq \frac{1}{256}(d-2)n^2(\ln \frac{n}{d} - O(1)).$$

The Ω bound follows from Theorem 7. □

5. Variations on the Reductions

This section describes two variations on the reductions presented in previous sections, either of which may conceivably lead to improved lower bounds on $U(d,n)$. Proofs in this section are deferred to the full paper [8].

5.1. t-Edge-Universal Sequences for the Cycle

This section describes a reduction that could conceivably improve the lower bound on $U(d,n)$ from $\Omega(dn^2 \log \frac{n}{d})$ to $\Omega(d^2n^2 \log \frac{n}{d})$. The basic idea is to use Lemma 5 in place of Lemma 6 in the proof of the reduction of Theorem 8.

Definition: Let $E(t,d,n)$ be the length of the shortest sequence that is t-edge-universal for $\mathcal{G}(d,n)$.

Note that $C(t,n) \geq E(t,2,n)$.

Theorem 18: Let $n-d-1$ be a multiple of $2(d-1)$. Then

$$U(d,n) \geq \frac{d}{2}E((d-1)s,2,m),$$

where $s = \frac{(d-2)(n-d-1)}{4} + 1$ and $m = \frac{n-d-1}{2(d-1)}$.

For instance, suppose it could be proven that $E(t,2,m) = \Omega(tm \log m)$, a generalization of Theorem 16. Then for $3 \leq d = o(n)$, it would follow from Theorems 7 and 18 that $U(d,n) = \Omega(d(ds)m \log m) = \Omega(d^2n^2 \log \frac{n}{d})$.

5.2. Commuting Sequences for Arbitrary Graphs

This section generalizes the notion of circumnavigations to graphs other than cycles, and shows how a lower bound on this generalization would also yield a lower bound on $U(d,n)$.

For any $G \in \mathcal{G}(d,n)$, any start vertex v_0 of G, and any $U \in \{0,1,\ldots,d-1\}^*$, let $(v_0, v_1, \ldots, v_k)$

be the sequence of vertices traversed by U when started at v_0. For any two vertices u and w of G, such a sequence U is said to *commute between u and w t times* starting at v_0 if and only if there exist $0 \leq i_1 < i_2 < \cdots < i_{2t+1} < k$ such that $v_{i_{2j+1}} = u$ for $0 \leq j \leq t$ and $v_{i_{2j}} = w$ for $1 \leq j \leq t$. U is a *t-commuting sequence* for $\mathcal{G}(d,n)$ if and only if U commutes between each pair of vertices in each $G \in \mathcal{G}(d,n)$ t times starting at any vertex in G. $K(t,d,n)$ denotes the length of the shortest t-commuting sequence for $\mathcal{G}(d,n)$.

For example, note that $K(s,2,2m) \geq C(2s,m)$, since each commute between a fixed pair of antipodal vertices on a $2m$-cycle labeled $\beta\beta$ will cause two circumnavigations of the m-cycle labeled β. Thus, Theorem 8 is a corollary (for $d' = 2$) of the following theorem.

Theorem 19: Suppose $2 \leq d' \leq d$, and let n be a multiple of $8(d-d'+1)$. Then

$$U(d,n) \geq \frac{d}{d'} K(s-1,d',m),$$

where $s = \frac{(d-2)n}{8} + 1$ and $m = \frac{n}{4(d-d'+1)}$.

6. Open Problems

There are many interesting open problems suggested by this work. Perhaps the most important is to try to extend these lower bounds to a time-space tradeoff for undirected graph connectivity, using the model suggested in Section 1. The first goal would be to prove that, for constant degree and constant number of pebbles, any automaton requires time $\Omega(n^2)$ to traverse n-vertex graphs, since we now know this to be true for one pebble. Beame *et al.* [6] have recently shown a lower bound of $\Omega(n \log n)$ for one variant of this model. Their argument is based on a different universal traversal sequence lower bound (Sipser and Szemerédi [personal communication]), and doesn't seem to extend using the one in Section 2.

Another open problem is to improve this paper's lower bound so that it is closer to the known upper bounds given in Table 1 in Section 1. In particular, it would be rewarding to prove a lower bound of $\Omega(n^3)$ for constant degree graphs. New techniques will be needed to accomplish this. The proof of Theorem 2 showed that, for a fixed, labeled graph H, a sequence that is universal just for the family $\{G_x\}$ must be long ($\Omega(d^2n^2)$). For this restricted problem, our bound is essentially tight: there are many graphs H such that the family $\{G_x\}$ has a universal traversal sequence of length $O(d^2n^2)$. Thus a better lower bound will require consideration of a larger family of graphs and/or labelings, and yet we are able to prove directly only weaker lower bounds for larger classes of graphs.

It would also be interesting to extend our lower bound to labeled cycles (the case $d = 2$), for which the lower bound $\Omega(n \log n)$ is known to hold (Bar-Noy *et al.* [5]), but $\Omega(n^2)$ is not. If the latter could be established and then generalized to multiple circumnavigations, it would yield the lower bound $\Omega(n^3)$ for any $d \geq 3$, using the reduction of Section 4.

Extending the $\Omega(d^2n^2)$ bound to values of d closer to $n/2$ would also be enlightening, particularly since a recent result of Chandra *et al.* [11] yields an upper bound of $O(n^3 \log n)$ for all $d \geq n/2$. This, together with our lower bound, shows that $U(d,n)$ is not monotone in d, but it is not yet known whether $U(d,n)$ drops sharply at $d = n/2$, as does the expected cover time of a random walk [11]. It is also not known whether $U(d,n)$ is monotone in d up to some threshold, perhaps $n/2$. There is also a gap in our knowledge for $d \geq n/2$. The best lower bound for $d \geq n/2$ is $\Omega(n^2/\log n)$ (Alon and Ravid [3]), well below the known upper bound of $O(n^3 \log n)$ [11].

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