# CSC375 IP for scheduling problem

Question 3 of term test 3 asks for a $\{0,1\}$ IP formulation of a scheduling problem considered in problem set 2 and term test 2. Namely, we are given $n$ jobs $J_1, \ldots, J_n$ with $J_i = (d_i, t_i, v_i)$ where $d_i$ is the deadline for job $J_i$, $t_i$ is its processing time, and $v_i$ is its profit. Assume all input parameters are positive integers. A schedule is a function $\sigma : \{1, \ldots, n\} \rightarrow \{0, 1, 2, \ldots\} \cup \{\infty\}$ where $\sigma(i) = \infty$ means that job $J_i$ is not scheduled and $\sigma(i) = k$ means that job $J_i$ begins executing at time $k$. A schedule is feasible if
(1) for all $i \neq j$ if $\sigma(i) \neq \infty$ and $\sigma(j) \neq \infty$ then $[\sigma(i), \sigma(i) + t_i) \cap [\sigma(j), \sigma(j) + t_j) = \emptyset$
(2) for all $i$, if $\sigma(i) \neq \infty$ then $\sigma(i) + t_i \leq d_i$.
That is, no two scheduled jobs will overlap and every scheduled job finishes before its deadline. The optimization problem is to find a feasible schedule $\sigma$ that maximizes $\sum_{\sigma(i) \neq \infty} v_i$; that is, to maximize the profit of scheduled jobs in a feasible schedule.

We will introduce $\{0,1\}$ variables $x_i$ with the intended meaning that $x_i = 1$ if and only if job $J_i$ is scheduled. We assume that the jobs have been sorted so that $d_1 \leq d_2 \ldots \leq d_n$ and exploit the fact that a subset of jobs $S$ can be feasibly scheduled if and only if the jobs in $S$ can be scheduled in the order of their deadlines (and without any gaps). That is, $\sigma(i) < \sigma(j) \Rightarrow d_i \leq d_j$.

The following IP is a formulation of this optimizatiuon problem:

$max \sum_{1 \leq i \leq n} x_i v_i$
subject to:
$\sum_{1 \leq j \leq i} x_j t_j \leq d_i$ for $1 \leq i \leq n$
$x_i \in \{0, 1\}$

If $x_1, x_2 \ldots x_n$ is a feasible IP solution then $\sigma(i) = \sum_{1 \leq j \leq i-1} x_j t_j$ if $x_i = 1$ and $\sigma(i) = \infty$ if $x_i = 0$ is a feasible schedule.