Due: Wednesday, April 5, beginning of lecture

NOTE: Each problem set only counts 5% of your mark, but it is important to do your own work (but see below). Similar questions will appear on the first term test which will cover material relating to both assignment 1 and assignment 2. You may consult with others concerning the general approach for solving problems on assignments, but you must write up all solutions entirely on your own. However, for problem set 1, you may work in pairs for the bonus questions. Anything else is *plagiarism*, and is subject to the University's Code of Behavior. You will receive 1/5 points for any (non bonus) question/subquestion for which you say "I do not know how to answer this question". You will receive .5/5 points if you just leave the question blank.

1. Suppose we are given a unit capacity flow network $\mathcal{F}$; that is a network in which all capacities c_e are in $\{0, 1\}$. Show that Dinic's algorithm terminates in $O(n^{2/3})$ blocking steps where n is the number of nodes in the network.
   Hint: Use the fact that max flow = min cut in considering the levelled graph.
   [20 points]

2. Consider the problem of maximizing the number of clauses satisfied by a (exact) 2-CNF formula F; that is, there are precisely two literals per clause. Consider the following local-search algorithm:

   Set $\tau(x_i) = true$ for all variables x_i.
   While there is a variable x_i for which complementing x_i will result in an increase in the number of clauses satisfied
       Complement $\tau(x_i)$
   End While

   We want to show that any local optimum τ (produced by the local search algorithm) is a 2/3 approximation (or 3/2 if you like approximation ratios to be ≥ 1). In fact, show that the number of clauses satisfied by τ is at least $(2/3) * m$ where m is the number of clauses in F.

   Hint: For each i, consider the number N_i^0 (respectively N_i^1) of clauses C_j that contain the variable x_i and C_j is not satisfied by τ (respectively exactly 1 literal is satisfied by τ and that literal is the one satisfied by $\tau(x_i)$). Argue that $N_i^0 \leq N_i^1$ for all i and then sum for all i. You want to bound (by 1/3) the ratio of the total number of unsatisfied clauses divided by the total number of clauses.
   [20 points]

3. Consider the following *3 to 2 frequency set cover problem*: We are given a collection of sets $\mathcal{S} = \{S_1, \ldots, S_m\}$ for $S_i \subseteq U$ with the property that every $u \in U$ occurs in exactly three different sets S_i in $\mathcal{S}$. There is also a cost function $c : \mathcal{S} \rightarrow \Re^{\geq 0}$ and we let c_i denote the cost of set S_i. A feasible solution is a sub-collection $\mathcal{S}' \subseteq \mathcal{S}$ such that every $u \in U$ occurs in at least two different sets S_i in $\mathcal{S}'$. The goal is to find a feasible solution $\mathcal{S}'$ so as to minimize the cost $c(\mathcal{S}') = \sum_{S_i \in \mathcal{S}'} c_i$.

   (a) Formulate the *3 to 2 frequency set cover problem* as a $\{0, 1\}$ IP [10 points]

(b) Show how to use LP relaxation + rounding to obtain an 2-approximation algorithm. Explain why your rounded solution is a feasible solution to the IP and why it provides a 2-approximation. [10 points]

4. Consider the following *graph triangle 6 colouring problem.* Given an edge weighted graph $G = (V, E)$ with edge weight $w(e) \geq 0$ on each edge $e \in E$, the goal is to find a 6-colouring (of the nodes) $\sigma : V \rightarrow \{1, 2, \ldots, 6\}$ so as to maximize the value $val(\sigma, G, w)$ of a colouring which will now be defined. A triangle $T(x, y, z)$ is a set of three nodes $\{x, y, z\}$ such that $(x, y), (y, z), (x, z) \in E$. A triangle $T(x, y, z)$ is properly coloured by σ if $|\{\sigma(x), \sigma(y), \sigma(z)\}| = 3$; that is, the three vertices get different colours. The value of a colouring $val(\sigma, G, w) = \sum_{T(x,y,z) \text{ is properly coloured}} [w(x, y) + w(y, z) + w(x, z)]$.

   (a) Provide a randomized algorithm for computing a 6-colouring σ such that for all inputs (G, w), the expected value $E[val(\sigma, G, w)] > \frac{1}{2} val(\sigma^*, G, w)$ where σ^* is an optimal 6-colouring for this problem. Justify the bound on the expectation. [15 points]

   (b) Explain how you can apply the method of conditional expectations to deterministically compute a 6-colouring such that $val(\sigma, G, w) > \frac{1}{2} val(\sigma^*, G, w)$ [5 points]

5. Consider the $Exact - max - 3SAT$ problem and the naive randomized algorithm for which we know $E[weight\ of\ satisfied\ clauses] \geq \frac{7}{8} \sum_{1 \leq i \leq r} w_i \geq \frac{7}{8} OPT$ for a formula $F = C_1 \wedge C_2 \ldots \wedge C_r$. De-randomization by the method of conditional expectations gives us a deterministic $\frac{7}{8}$ approximation algorithm. The purpose of this question is to show that this de-randomized algorithm can be viewed as a greedy (i.e. "priority") algorithm in the following sense. Let the input items be the propositional variables $x_1, \ldots, x_n$ of F where each variable x_i is represented by its index i, the indices of the clauses $\{C_j^+\}$ in which x_i appears positively, and the indices of the clauses $\{C_j^-\}$ in in which x_i appears negatively. Then without needing any sorting of the input items, show how to iteratively make an irrevocable decision about each x_i. [10 points]