Due: Wed, October 6, beginning of lecture

NOTE: Each problem set only counts 5% of your mark, but it is important to do your own work (but see below). Similar questions will appear on the first term test. You may consult with others concerning the general approach for solving problems on assignments, but you must write up all solutions entirely on your own. Anything else is *plagiarism*, and is subject to the University's Code of Behavior. You will receive 1/5 points for any (non bonus) question/subquestion for which you say "I do not know how to answer this question". You will receive .5/5 points if you just leave the question blank.
Advice: Do NOT spend an excessive amount of time on any question and especially not on a bonus question. If you wish to spend "free time" thinking about (say) bonus questions that is fine but you should not sacrifice time needed for other courses.

1. (20 points)

   Let $G = (V, E)$ be a chordal graph with perfect elimination order $v_1, \ldots, v_n$. Show that the following greedy algorithm will compute a maximum size independent set in G:

   $S := \emptyset$
   For $i = 1..n$
       If v_i is not adjacent to any $v \in S$, then $S := S \cup \{v_i\}$
   End For

2. (25 points)

   Consider the unweighted and weighted MIS problem for graphs induced by the intersection of axis parallel rectangles in the plane having a fixed height-length ratio. That is given n axis parallel rectangles R_i with dimensions $h_i \times c \cdot h_i$ for some fixed c, consider the induced graph $G = (V, E)$ where $V = \{R_i | i = 1, ..., n\}$ and $(R_i, R_j) \in E$ iff R_i and R_j intersect.

   (a) (5 points) Show that this class of graphs are not k clawfree for any fixed k.

   (b) (5 points) For the case when all rectangles have the same dimension $h \times w$, show that this class of graphs is 5-clawfree.

   (c) (5 points) Design a greedy algorithm for the weighted MIS problem which has a 4-approximation ratio when all the rectangles have the same dimension $h \times w$. Justify the approximation ratio.

   (d) (10 points) Design a greedy algorithm for the unweighted case which has a 4-approximation ratio when the dimensions $h_i \times c \cdot h_i$ are arbitrary. Justify the approximation ratio.

3. (10 points)

   Consider the problem of computing a weighted maximum matching in a bipartite graph. Show that this problem is an instance of matroid intersection, namely computing the maximum weight of a set of elements that are independent in matroids $\mathcal{M}_1$ and $\mathcal{M}_2$.

4. This question concerns Dijkstra's least cost path algorithm and variants. (30 points)

   (a) (10 points) Give an inductive proof that Dijsktra's algorithm optimally computes the least cost paths from a given source s to all other nodes when all edges have positive costs.

   (b) (5 points) Show that the algorithm will not always compute optimal paths when there are negative edges.

   (c) (10 points) Consider a variant of the shortest path problem (from a given source s to all ther nodes) where the cost of a path π is now defined to be $cost(\pi) = \max_{e \in \pi} c(e)$; that is, the cost of π is the maximum cost on any of its edges. Show how to modify Dijsktra's algorithm (and its proof of optimality) so as to optimally compute least cost paths for this new cost measure. Would your algorithm work if there were negative cost edges for this new cost measure?

   (d) (5 points) Can you suggest a property of a cost function $cost(\pi)$ such that a simple modification (i.e. replacing the standard sum of edge costs measure by the new cost measure) of Dijkstra's algorithm will compute optimal cost paths?