Due: Wed, October 3, beginning of lecture

NOTE: Each problem set only counts 5% of your mark, but it is important to do your own work (but see below). Similar questions will appear on the first term test. You may consult with others concerning the general approach for solving problems on assignments, but you must write up all solutions entirely on your own. Anything else is *plagiarism*, and is subject to the University's Code of Behavior. You will receive 1/5 points for any (non bonus) question/subquestion for which you say "I do not know how to answer this question". You will receive .5/5 points if you just leave the question blankxa.
Advice: Do NOT spend an excessive amount of time on any question and especially not on a bonus question. If you wish to spend "free time" thinking about (say) bonus questions that is fine but you should not sacrifice time needed for other courses.

1. (a) Design a greedy algorithm for the m-machine interval scheduling algorithm. [10 points]

   (b) Use an inductive argument to show that after the i^{th} iteration of the algorithm ($i \geq 0$), the partial solution $\sigma_i : \{1, \ldots, i\} \rightarrow \{0, 1, \ldots, m\}$ is promising and hence proving that the algorithm is optimal.
   Hint: In one of the cases for the induction step, the exchange will involve more than one interval. [10 points]

   (c) Bonus question: Can you prove the optimality of your algorithm using a "charging argument". Hint: I don't know the answer to this question. [15 points]

2. (a) Let EFT-colour denote the greedy colouring algorithm that first sorts so that $f_1 \leq f_2 ... \leq f_n$ and then greedily colours using the lowest non-conflicting colour. Give an input instance for which EFT-colour is not optimal. [10 points]

   (b) Bonus question: The student who obtains the best lower bound on the approximation ratio for EFT-colour (ie exhibits an input instance with the worst ratio) will receive an additional 10 points. If (essentially) the same input instance is obtained by more than one person that these bonus marks will be split equally.

   (c) Bonus question: Can you derive an upper bound on the approximation ratio for EFT-colour; that is, an upper bound on a constant c such that EFT-colour$[\mathcal{I}] \leq c * OPT[\mathcal{I}]$ for every input set $\mathcal{I}$. [15 points]

   (d) Exhibit a graph which is not a perfect graph. [5 points]

3. This question concerns the computation of a MST. Let $G = (V, E)$ be a connected graph with edge weights $\{c(e) | e \in E\}$.

   (a) Suppose all the edge weights are distinct and let $v \in V$ be an arbitrary vertex. Prove or disprove the following statement: The MST must contain at least one minimum cost edge adjacent to v. [10 points]

(b) Suppose $c(e_1) < c(e_2) < c(e_3) \ldots < c_{m-1} = c_m$. That is, all edge weights are distinct except for the largest two edge weights which are equal. Prove or disprove the following statement: The MST is unique. [10 points]

4. We wish to prove the following result: Consider a set of symbols $a, b, c, d, \ldots$ with probabilities $f_a, f_b, \ldots.$ summing to 1 where at least one probability is strictly great than $\frac{2}{5}$. Show that the optimal Huffman prefix tree contains a leaf at distance 1 from the root.

   (a) Show that this result is true when there are exatly four symbols. Note that the result is trivial when there are two or three symbols. [10 points]

   (b) Argue that this result holds when there are more than four symbols using the fact that it is true for four symbols. [10 points]