CSC373S **Problem Set 3** **Spring, 2013**

Due: April 1, beginning of lecture; NOTE change of original date

NOTE: Each problem set only counts 5% of your mark, but it is important to do your own work (but see below). Similar questions will appear on the second term test. You may consult with others concerning the general approach for solving problems on assignments, but you must write up all solutions entirely on your own. Anything else is *plagiarism*, and is subject to the University's Code of Behavior. You will receive 1/5 points for any (non bonus) question/subquestion for which you say "I do not know how to answer this question". You will receive .5/5 points if you just leave the question blank. Advice: Do NOT spend an excessive amount of time on any question and especially not on a bonus question. If you wish to spend "free time" thinking about (say) bonus questions that is fine but you should not sacrifice time needed for other courses.

1. (15 points)

   Consider the carpet production problem in section 7.1.2 of the text. Explain the objective function; that is, explain each of the coefficients preceding the 5 summations in the linear objective.

2. (15 points)

   Consider the "{0,1} IP" decision problem; namely, given a set of linear constraints, is there a {0,1} setting of the IP variables satisfying all the constraints. Show that the {0,1} IP decision problem is NP-complete. That is, for some known NP-hard or complete problem L, show that $L \leq_p IP$.

   Note: For the {0,1} case it is obvious that the problem is in NP. More generally, the IP decision problem is also in NP but this is not immediately obvious as we have to argue (using linear algebra) that if a set of linear constraints is satisfied by integers, then there is a solution where all the IP variables are not too big (i.e. have length polynomial in the length of the input representation).

3. (20 points)

   Consider the following partial vertex cover problem. We are given a graph $G = (V, E)$ which has both node weights $w : V \rightarrow \Re^{\geq 0}$ and edge penalties $p : E \rightarrow \Re^{\geq 0}$. Let $V' \subseteq V$ cover some set edges $E' \subseteq E$; that is, for every $e = (u, v) \in E'$, either u or v (or both) are in V'. The total cost of such a partial cover (V', E') is the sum of node weights in V' plus the sum of edge penalties for edges not in E'.

   (a) Express the partial vertex cover problem as an IP.

   (b) Provide an LP relaxation and rounding that yields a constant approximation algorithm for computing the total cost of a partial vertex cover. You must justify the approximation bound.

4. (10 points) Consider the Max-Cut problem (as in lecture 22).

   Show that in some instance, it helps to use the neighbourhood $N_2(A, B)$. Specifically, give an unweighted graph having a locality ratio $\frac{1}{2}$ for the $N_1(A, B)$ neighbourhood but where the local search algorithm using neighbourhood $N_2(A, B)$ is optimal when applied to graph G.
   Hint: The graph G is a small graph.

5. (20 points) Prove the key observation in slide 11 of L23. Note: this observation leads to the non-oblivious $\frac{3}{4}$ locality ratio for Exact Max-2-Sat as suggested on slide 10.

6. (10 points) Consider a propositional formula F in exact 3 CNF form; that is every clause has exactly three literals involving 3 distinct variables. Using ideas from the naive randomized algorithm, show that F is satisfiable if F has at most 7 clauses.

7. (20 points) You are to generate a random 2CNF formula over 4 variables as follows: Consider the $\binom{4}{2} = 6$ ways to select 2 of the 4 variables. Now for each such choice (say x_1, x_2), randomly define two (different) clauses by setting each literal to be the variable (say x_1) or its complement ($\bar{x}_1$). This will result in 12 clauses, each with two literals. Randomly choose the order in which you will set variables and apply the method of conditional expectations to derive a truth assignment satisfying at least a fraction $\frac{3}{4} = 9$ clauses.

8. Consider the following variant of the Max-Sat problem:
   We are given a clause weighted CNF Formula $F = C_1 \wedge C_2 \ldots \wedge C_m$ with $w_j = w(C_j)$, the weight of clause C_j.
   Goal: to find a truth assignment that maximizes the weight of clauses that are satsified by at least two literals.

   (a) (10 points) Represent this problem as a {0,1} IP.

   Note: (Without loss of generality we can assume each clause has at least two literals but your formulation should work without this assumption.

   (b) **Bonus**

   We showed how to relax the IP for the standard Max-Sat problem. Relax the IP above and see what approximation ratio you can derive.
   Note: When I wrote down this problem, I thought there could be an easy modification of the analysis for the standard Max-Sat problem but that is not the case. It is instructive to see where the analysis becomes complicated in this variant.