1.

2.

3. Show how to polynomial time reduce the 3-colouring search problem to the 3-colouring decision problem. Hint: Introduce three new nodes to represent the three colours and then try to colour one node at a time.

Solution Sketch We first ask if $G$ can be 3-coloured; if not, we are done. So now we assume that $G$ can be 3-coloured. To actually find a colouring of $G$, the idea is (as in the hint) to colour one node at a time so that the partial solution at any time can be extended to a 3-colouring of all nodes in the given graph $G$. To that end, we introduce three nodes (call them $R$, $B$, $Y$) (for red, blue, yellow) and add edges between them to form a triangle and add that triangle to the given graph $G$. Let’s call the new graph $G'$. To colour a node $v$ we will add edges (to $G'$) between $v$ to exactly 2 of the nodes in $\{R, B, Y\}$. The intention is that if (for example) we add edges $(v, R)$ and $(v, Y)$, then we are colouring $v$ by $B$ (i.e. blue). As we consider each node $v$, we check (using the 3-colouring decision problem) if a suggested colouring of $v$ (by adding 2 edges as above) still makes it possible to colour $G'$ with 3 colours, coloured to any adjacent nodes. We do this by adding an edge to $R$. Then we will add edges to

4. Show that the following decision problems are NP-complete. To establish the NP hardness, you can use a polynomial time transformation from any problem in the tree of transformations on slide 14 of lecture 15.

(a) Half independent set (HIS) where
HIS = \{ $G|G = (V, E)$ and $G$ has an independent set of size at least $\lceil |V|/2 \rceil$ \}.

Solution Sketch. To simplify the notation let me assume that $|V|$ is even so we don’t have to worry about floors and ceilings. We will describe a transformation of $\text{IS} \leq_p \text{HIS}$ where $\text{IS} = \{(G, k)|G$ has an independent set of size at least $k\}$. We have three cases to consider depending on whether $k = |V|/2, k < |V|/2, k > |V|/2$. Let $n = |V|$.

- The first case $k = n/2$ is simple being the transformation $(G, k) \rightarrow G$.
- $k < 2$. We want to add an independent set of $r$ new nodes to $G$ forming a graph $G'$ with $n' = n + r$ nodes. We choose $r$ so that $k + r = (n + r)/2$. That is, we choose $r = n - 2k$. We claim that $G$ has a size $k$ independent set iff $G'$ has a size $n'/2$ independent set.
- $k > n/2$. We basically do a simple construction as in the previous case but now we add a size $r = n - 2k$ clique and connect each node in this clique to every node in the initial graph $G$. Alternatively, we can just add a $r$ clique and choose $r$ so that $k + 1 + r = (n + r)/2$.

(b)