CSC 373: Algorithm Design and Analysis
Lecture 3

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Lecture 3: Outline

- Write bigger and get better markers
- A little more on charging arguments
- Continue examples of greedy algorithms
  1. Maximal Matching algorithm for minimum vertex cover
  2. m machine interval scheduling (m-ISP)
  3. Interval colouring

Charging arguments

- A common method for proving optimality or approximation results for an optimization algorithm ALG is by a charging argument.

- For a profit maximization problem:
  1. Charge the profit of an arbitrary solution (and hence that of an optimal solution OPT) to the profit of your ALG.
  2. Argue that not too much profit from OPT gets charged to ALG.

- For a cost minimization problem:
  1. Charge the cost of ALG to an OPT solution
  2. Argue that not too much cost from ALG is charged to OPT.

Charging argument for EFT (as discussed last lecture)

- For the (unweighted) ISP problem, the profit of an algorithm is simply the number of intervals selected.
- For an input I, we will write
  - $|ALG(I)|$ to denote the profit of algorithm ALG
  - $|OPT(I)|$ to denote an optimal solution
- Then to show optimality of EFT for ISP, it suffices to show that

There is a 1-1 function h mapping $OPT(I)$ into $EFT(I)$.

- Since OPT is an optimal solution, the mapping must be onto.

Charging argument to obtain approximation bound

- As stated in the first class, I like to integrate some results about approximation algorithms as we proceed rather than treat approximation algorithms as a separate topic.
- We can easily adapt the EFT algorithm so as to apply to the JISP problem.
- **JISP problem:** Two intervals are compatible if
  - they do not intersect and
  - they do not belong to the same job class.

Claim (Question 2 of problem set)

For the JISP problem we can show that the same h is a 2-1 function mapping $OPT(I)$ into $EFT(I)$.

A greedy algorithm for the vertex cover problem

Vertex Cover problem

- **Given:** A graph $G = (V, E)$. A vertex cover for G is a subset $V' \subset V$ such that for every edge $e = (u, v) \in E$, at least one of u or v is in V'.
- The goal is to find a minimum size vertex cover.

Maximal Matching (MM) [greedy algorithm for vertex cover]

$V' := \varnothing$
While $E \neq \varnothing$
For any $e = (u, v) \in E$, $V' := V' \cup \{u, v\}$
%add both endpoints of the edge to the cover
Delete all edges from E that are adjacent to this edge e.
End While

The MM greedy algorithm is a 2-approximation for vertex cover

- Let $OPT(G)$ be any (e.g. an optimal) vertex cover and $MM(G)$ be the solution of algorithm MM for graph G.
- Then $MM(G)$ is a vertex cover for G and $|MM(G)| \leq 2 \cdot |OPT(G)|$ for all graphs G.

The m-ISP problem

- The m "machine" interval scheduling problem (m-ISP) schedules a set of intervals on m machines so that

intervals assigned to the same machine do not intersect.

- We next consider two extensions of the one machine EFT algorithm.

First Fit EFT

1: Sort intervals so that $f_1 \leq f_2 \leq \ldots \leq f_n$
2: **for** $j = 1$ to n **do**
3: $\quad k := \begin{cases} \min\{\ell : I(j) \text{ doesn't intersect intervals on machine } \ell\} & \text{if such } \ell \text{ exists} \\ 0 & \text{if no such } \ell \end{cases}$
4: $\quad \sigma(j) := k$
$\quad$ *% $\sigma(i)$ specifies if and on which machine interval $I(j)$ is scheduled*
5: **end for**

Fact

First Fit EFT is not an optimal algorithm

An optimal greedy algorithm for m-ISP

Another extension of the one machine EFT algorithm:

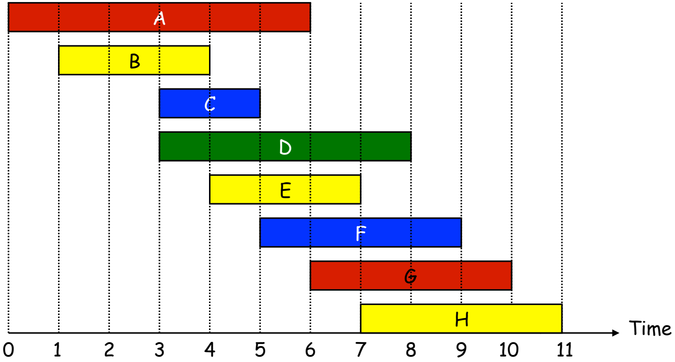
Best Fit EFT

Sort intervals so that $f_1 \leq f_2 \leq \ldots \leq f_n$
for $k = 1$ to m **do**
    $e_k = 0$
    % e_k specifies the lastest completion for intervals on machine k
end for
for $i = 1$ to n **do**
    Let $k := \begin{cases} \min\{\ell : s_i - e_\ell > 0\} & \text{if such } \ell \text{ exists} \\ 0 & \text{if no such } \ell \end{cases}$
    $\sigma(i) := k$
    % $\sigma(i)$ specifies if and on which machine interval $J(i)$ is scheduled
    $e_k := f_i$
end for

Interval colouring

Interval Colouring Problem

- Given a set of intervals, colour all intervals so that intervals having the same colour do not intersect
- **Goal:** minimize the number of colours used.



We use 4 colors in this example. **Question:** Is this optimal?

Interval colouring

Interval Colouring Problem

- Given a set of intervals, colour all intervals so that intervals having the same colour do not intersect
- **Goal:** minimize the number of colours used.

- We could simply apply the m-machine ISP for increasing m until we found the smallest m that is sufficient.

- **Note:** This is a simple example of a polynomial time reduction which is an essential concept when we study NP-completeness.

- **Note:** There are examples of graph classes where the colouring problem can be efficiently computed whereas on this class of graohs, the m-ISP problem is NP-hard when m is a parameter of the problem.

Greedy interval colouring

- Consider the EST (earliest starting time) for interval colouring.
  - Sort the intervals by non decreasing starting times
  - Assign each interval the smallest numbered colour that is feasible given the intervals already coloured.
- Recall that EST is a terrible algorithm for ISP.
- **Note:** this algorithm is equivalent to LFT (latest finishing time first).

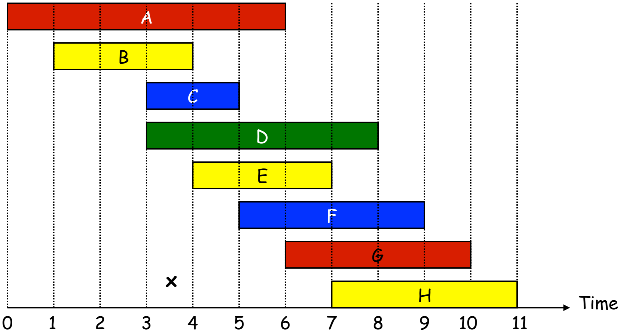
Theorem

EST is optimal for interval colouring

Greedy Interval Colouring

```
Sort intervals so that $s_1 \leq s_2 \leq \ldots \leq s_n$
for $i = 1$ to $n$ do
    $k := \min\{\ell : \ell \neq \chi(j)$ for all $j < i$ such that the $j^{th}$ interval intersects the $i^{th}$ interval$\}$
    $\sigma(i) := k$
    /* the $i^{th}$ interval is greedily coloured by the smallest non conflicting colour */
end for
```

An example of interval colouring



- We use the colors in the following order: red, yellow, blue, green

Idea of optimality proof

Look at the interval that (first) caused the largest colour, say k, to be used by EST. Then there must be k intervals containing a given "time" x and hence k colours are required.