CSC 373: Algorithm Design and Analysis
Lecture 28

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Announcements and Outline

Announcements

- Announcement of CS Townhall (see next slide)
- Problem set 3 due on Monday, April 1. Note: I have made the second part of question 8 a bonus question.
- Term test 3 on Wednesday, April 3
- Plan to have final class wrapping up course on Friday, April 5.

Today's outline

- Continue local search for MIS and weighted MIS on $k+1$ clawfree graphs
- Recalling interval scheduling and the JISP problem.
- Chordal graphs (see slides 15 and 16 in Lecture 4)
- JISP (see slide 10 in Lecture 2 and Question 2 in Problem Set 1)
- Inductive k Independence Graphs

Townhall announcement

Good afternoon CS students,
This Thursday, March 28th at 11:00am, in BA3200, the CS department will hold a townhall for all Undergraduates enrolled in the CS POSt as well as first-years in the CS Stream.
In the coming days, we'll be sending you a quick survey, but Thursday's townhall meeting is your opportunity to let us know how we are doing, to give feedback on the program, and to tell us about your experience as an Undergraduate in Computer Science. We want to hear it all... the good, the bad and the ugly. And we want to hear from students in all years, first-years through to fourth.
It is the end of term, and we know that you are all very busy, so take a break from those assignments, come share your opinions, and have some lunch on us. There will be pizza! To make sure there is enough pizza for you, please RSVP (by 5:00pm Wednesday, 27th March). To do so, just reply to this email or send a message to `ugevent@cs.toronto.edu`
Many thanks, and I hope to see all of you on Thursday.
Karen Reid

Local Search for MIS on $k+1$ clawfree graphs

- Can we do better than a k-approximation for (W)MIS on $k+1$ clawfree graphs and hence for the (weighted) k set packing problem?

Local Search for MIS on $k+1$ clawfree graphs

- Can we do better than a k-approximation for (W)MIS on $k+1$ clawfree graphs and hence for the (weighted) k set packing problem?
- Khanna et al and Yu and Goldschmidt showed that we can get $\frac{k+1}{2}$ approximation for unweighted MIS on $k+1$ clawfree graphs using a "2-improvement" (i.e. essentially a Hamming distance 3) local search.
  - We can achieve $\frac{k}{2}+\epsilon$ approximation using t-improvement local search.
- In the weighted case, the locality gap for the oblivious distance t local search algorithm is $(k-1+\frac{1}{t})$ for weighted k-set packing.
- We will, however, soon see that we can improve upon the approximation ratio by either using a greedy initial solution and oblivious local search, or by using a non-oblivious local search.

The 2-improvement local search algorithm for unweighted MIS

- Define

$$Nbhd(S) = \Big\{ S \cup \{u, v\} \setminus \big(N(u) \cup N(v)\big) \mid u, v \text{ in the universe } U \Big\}$$

- Note that u and v don't have to be distinct. In other words, we can have $u = v$.

- And for the unweighted case, it must be that $|N(u) \cup N(v)| \leq 1$ to be useful in the local search.

Algorithm for unweighted MIS

1: $S := \varnothing$
2: **while** $\exists S' \in Nbhd(S)$ such that $|S'| > |S|$ **do**
3: $\quad S := S'$
4: **end while**

Halldorsson's Analysis of 2-improvement algorithm

Theorem

The 2-improvement algorithm for unweighted MIS is a $\frac{k+1}{2}$ approximation.

- The proof is essentially a counting (i.e. charging) argument
- Let A be the independent set produced by the greedy algorithm, and B be a maximum independent set.
- Let $A' = A \setminus (A \cap B)$ and $B' = B \setminus (A \cap B)$.
- Given the nature of the charging argument, we can assume that $A \cap B = \emptyset$ to simplify the argument.
- Let $B_1 = \{v \in B' \mid v \text{ has exactly one neighbor in } A\}$
- Let $B_2 = \{v \in B' \mid v \text{ has at least two neighbors in } A\}$
- Let $A_1 = \{u \in A' \mid u \text{ has a neighbor in } B_1\}$
- By the defintion of B_1, we have $|B_1| \geq |A_1|$.

Analysis of 2-improvement algorithm continued

- If $|B_1| > |A_1|$, then by the pigeonhole principle, $\exists x \in A_1$ such that at least two y_i in B_1 are adjacent to x. Thus, A is not 2-local optimum.
- So local optimality shows that $|B_1| = |A_1|$.
- Next we note that $|B_1| + 2 \cdot |B_2| \leq$ [number of edges between A' and B'] $\leq k \cdot |A'|$ since the graph is k+1 clawfree.
- Therefore adding $|B_1| = |A_1|$ to both sides of the inequality, we get $2 \cdot |B_1| + 2 \cdot |B_2| \leq (k+1) \cdot |A|$,
- Hence $2 \cdot |B| \leq (k+1) \cdot |A|$ showing that the algorithm is a $\frac{k+1}{2}$ approximation.
- This differs from the local search approach to Max-Cut where bigger neighbourhood sizes do not essentially help apprxoximation ratio.
  - Here swapping in 2 elements can substantially help (reducing the approximation from k to $\frac{k+1}{2}$).
  - But then further improvements do not substantially help (will only reduce approximation to $\frac{k}{2} + \epsilon$).

Approximation improvements for weighted MIS in $k+1$ clawfree graphs

- Chandra and Halldorsson show a $\frac{2(k+1)}{3}$ approximation.
- They use a greedy algorithm to initially approximate a max weight independent set.
- Then they use local search using a neighbourhood determined by a claw C with center in S and talons not in S such that the talons will improve the solution.
- Let the neighbourhood of C in S be

$$N(C,S) = \left\{ v \in S \mid \exists u \in C \text{ such that } (u,v) \text{ an edge} \right\}$$

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while ∃ claw C such that w(S ∪ C \ N(C,S)) > w(S) do
    S := (S ∪ C) \ N(C,S)
end while
```

Berman's non oblivious local search

- Berman uses a non-oblivious local search algorithm where the potential function is $w^{(2)}(S) = \sum_{v_i \in S} w_i^2$:

Let S be a maximal independent set
while $\exists$ claw C such that $w^{(2)}(S \cup C \setminus N(C,S)) > w^{(2)}(S)$ **do**
    $S := (S \cup C) \setminus N(C,S)$
end while

- Consider the motivation for such a non-oblivious potential function!

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2: **while** $\exists$ claw C such that $w^{(2)}(S \cup C \setminus N(C,S)) > w^{(2)}(S)$ **do**
3: $S := (S \cup C) \setminus N(C,S)$
4: **end while**

- Consider the motivation for such a non-oblivious potential function!

- The motivation is that given two independent sets, one with relatively few elements of large weight, and one with many elements of small weight, which is "better"?

Some further comments on $k+1$ clawfree graphs

- All of the examples given so far for classes of $k+1$ clawfree graphs would also satisfy the definition of locally VCC_k graphs.
- However, for $k \geq 3$, deciding if a graph G is in VCC_k is an NP complete problem. This is easy to see when we realize that G is in VCC_k iff the complement graph $\bar{G}$ is k colourable.
- For any fixed k, we can determine if G is $k+1$ clawfree can be done in polynomial time by checking the neighbourhood of each vertex.
- Such an exhaustive procedure would require say kn^{k+2} and would be prohibitive for large k. Complexity assumptions imply that we cannot expect to do much better.
- But for specific classes like the ones indicated, it is usually easy to determine the smallest k for which a given graph is $k+1$ clawfree.
- The class of 3 clawfree graphs is simply called clawfree and there are many results known for this special class.

Returning to interval scheduling and JISP

- While the intersection graph of unit intervals is 3 clawfree, we have already noted that, in general interval graphs (i.e. the intersection graph of arbitrary length intervals) is not $k + 1$ clawfree for any k.
- Yet we know that in the unweighted case we can do interval scheduling (i.e. the MIS problem for interval graphs) by a simple greedy algorithm that first sorts the intervals by non-decreasing finishing time and then accepts greedily.
- Interval graphs are an example of chordal graphs which can be characterized by the existence of a perfect elimination ordering (PEO).
- Namely, $G = (V, E)$ is a chordal graph iff and only if there is an ordering of the vertices $v_1, v_2, \ldots, v_n$ such that the *inductive neighbourhood* of $v_i = \{v_j \mid j > i \text{ and } (v_i, v_j) \in E\}$ is a clique or equivalently its inductive neighbourhood has independence number 1.
- For interval scheduling (i.e. interval graphs), the ordering $f_1 \leq f_2 \leq \ldots \leq f_n$ is a PEO.

Greedy algorithms for MIS on chordal graphs, JISP and inductive k-independence graphs

- In the unweighted case, the MIS problem for chordal graphs can be computed optimally by the greedy algorithm using the PEO. Many different proofs of this fact, one proof being by a charging argument.
- We also saw that the same algorithm would yield a 2-approximation for the JISP problem. (This was question 2 of Problem Set 1 and again a charging argument could be used for the proof.)
- We can generalize the PEO concept so as to model the JISP problem. Namely, (making up some terminology), let us say that G has a k-PEO if there is an ordering of the vertices $v_1, v_2, \ldots, v_n$ such that the *inductive neighbourhood* of $v_i = \{v_j \mid j > i \text{ and } (v_i, v_j) \in E\}$ has independence number k.
- A graph having a k-PEO will be called an inductive k-independence graph
- That is, chordal graphs have a 1-PEO which is the same as a PEO.

JISP induced graphs

- It is not hard to see that the graphs induced by the JISP problem are inductive 2-independence graphs using the same ordering by non-decreasing finishing times.
- In fact, the inductive neighbourhoods have clique cover number 2 since neighbours v_j of v_i (with $j > i$) will either intersect at the finishing time f_i of vertex v_i (i.e. the i^{th} interval) and/or will belong to the same job class κ_i.
- The same greedy algorithm (sort by k-PEO and accept greedily) will yield a k-approximation for the MIS problem on any inductive k-independence graph and hence a 2 approximation for the JISP problem.

Other examples of inductive k-independence graphs for small k

- The intersection graphs of unit discs are inductive 3-independence graphs.
  What is the 3-PEO?

- The intersection graphs of unit squares are inductive 2-independence graphs.
  What is the 2-PEO?

- The intersection graphs of arbitrary radius discs are inductive 5-independence graphs.
  What is the 5-PEO?

- Note that every $k+1$ clawfree graph is an inductive k-independence graph.