CSC 373: Algorithm Design and Analysis
Lecture 27

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Announcements and Outline

Announcements

- Problem set 3 is now complete and due April 1.
- Term Test 3 in lecture, Wednesday, April 3.
- Tutorial today. Material in tutorials and lectures is relevant to the upcoming term test and final exam.

Today's outline

- Continue discussion of set packing
- The k-set packing problem
- $k+1$ clawfree graphs
- Greedy and local search algorithms for set packing and MIS on $k+1$ clawfree graphs
- Charging arguments

Recall the weighted set packing problem

The weighted set *packing* problem

- As in the set cover problem, we are given a collection of sets $\mathcal{C} = \{S_1,\ S_2, \ldots, S_n\}$ over a universe $U = \{e_1,\ e_2, \ldots, e_m\}$ and $w_i = w(S_i)$.
- Goal: Choose a subcollection $\mathcal{C}'$ of disjoint sets so as to *maximize* $\sum_{i:S_i \in C'} w_i$.
- When the size of the sets S_i is restricted such that $|S_i| \leq k$ we call this the k-set packing problem.

Set packing (SP) as an MIS problem

- The graph theoretic interpretation of the above problem is as follows:
  - Given a set packing instance define graph $G = (V, E)$ with $V = \{S_1,\ S_2,\ \dots S_n\}$ and $E = \{(S_i, S_j) \mid S_i \bigcap S_j \neq \varnothing\}$.
  - The (weighted) set packing problem becomes the (weighted) maximum independent set (W)MIS problem on this graph.

- **Note:** This is another example of a polynomial time transformation:

$$\text{SP} \leq_p \text{MIS}.$$

- Given an MIS problem, interpreting it as a set packing problem is also quite straightforward.
  - The set of elements $U = e_1,\ e_2,\ \dots, e_m$ consist of the edges of the graph $G = (V, E)$.
  - The collection of sets S_i for $1 \leq i \leq |V|(= n)$ is given by the adjacency list of the edges.
  - Note that in this case, $m \leq n^2$, $|S_i| \leq n$ and $|S_i \cap S_j| \leq 1$.

- That is, this second transformation shows that MIS $\leq_p$ SP.

Complexity of SP and MIS

- As mentioned , some experts believe BPP (and hence ZPP and RP) are the same as P. But in any case, it is strongly believed that NP $\neq$ RP.
- In arbitrary graphs MIS is hard to approximate to within a factor of $n^{1-\epsilon}$ for any $\epsilon > 0$ (assuming ZPP $\neq$ NP), while it trivial to get an approximation factor of n.
- For (weighted) SP, it is also immediate to obtain an approximation of $\min\{n, m\}$.
- The transformation MIS $\leq_p$ SP shows that under the same complexity assumption (ZPP $\neq$ NP) that it is hard to approximate set packing to within a factor $\min\{n, m^{\frac{1}{2}-\epsilon}\}$ for any $\epsilon > 0$.

Greedy algorithms for set packing

- It is not difficult to show that a natural greedy algorithm obtains an approximation ratio of $\min(n, m)$ in the case of the weighted set packing problem. What would you try?

Greedy algorithms for set packing

- It is not difficult to show that a natural greedy algorithm obtains an approximation ratio of $\min(n, m)$ in the case of the weighted set packing problem. What would you try?
  - Perhaps the most basic and natural algorithm is to sort sets so that $w(S_1) \geq w(S_2) \geq \ldots \geq w(S_n)$ and accept greedily (i.e. take the set if it does not conflict).
  - Another natural greedy algorithm for the Weighted Set Packing Problem to sort the sets according to the ratio $w(S)/|S|$.
- For the set packing problem, these natural greedy algorithms have approximation ratio of $\min(n, m)$ which in practice is a very poor approximation ratio. We consider another variant.

A better set packing greedy algorithm

Sort sets according to the ratio $w(S)/\sqrt{|S|}$ and accept greedily.

- This variant can be shown to have an approximation ratio of $min(n, \sqrt{m})$ and by our complexity assumptions is essentially the best worst case approximation.

The k-Set packing problem

- We now take a closer look at the k-set packing problem and the graphs induced by it.
- k-set packing is a reasonable and practical restriction given the appllication to say auctions.
- In the graph induced by the set packing problem, the neighborhood of a set S_i is given by $N(S_i) = \{S_j \mid (S_i, S_j) \in E\}$ and since every set contains at most k elements, the neighborhood is vertex covered by at most k cliques.
- Such graphs are called locally-VCC_k (vertex clique cover) graphs.
- These are not the same as VCC_k graphs where the entire graph can be vertex covered by at most k cliques.

$k+1$ clawfree graphs

- Locally-VCC_k graphs belong to a broader class of graphs called $k+1$ clawfree graphs or locally-IS_k (independent set) graphs.
- G is said to be $k+1$ clawfree if for any vertex v in the graph, $N(v)$ has at most k independent vertices. That is, G does not contain a "claw" $K_{1,k+1}$, the complete bipartite graph, as an induced subgraph.
- Such graphs occur in various other scenarios as well.
- Intersection graphs of unit discs can be shown to be 6-clawfree. (Such graphs are sometimes studied in the context of wireless networks.)
- Unit interval graphs are 3-clawfree.
- Intersection graphs of axis parallel unit squares are 5-clawfree graphs.
- Note: However, intersection graphs of arbitrary size discs, intervals, squares are not $k+1$ clawfree for any k.

The MIS problem for $k+1$ clawfree graphs

Theorem

The natural greedy algorithm (sort by weights) is a k-approximation algorithm for WMIS problem on any $(k+1)$ clawfree graph.

- We will prove this theorem using a charging argument.
- Let C_{opt} represent an optimal set of vertices and let C_{gre} represent the the set of vertices obtained using the natural greedy algorithm.
- Let h be a mapping from C_{opt} to C_{gre}. For $\nu \in C_{opt}$, define $h(\nu)$:

$$h(\nu) = \arg \max_{v' \in C_{gre} : (v, v') \in E} w(v') \tag{1}$$

We assume that any ties are broken lexicographically.

Charging argument continued

- We first consider the unweighted case.
- We observe that there at most k vertices $\nu \in C_{opt}$ that can get mapped to the same $v' \in C_{gre}$ by the assumed clawfree nature of the graph. Why?

Charging argument continued

- We first consider the unweighted case.
- We observe that there at most k vertices $\nu \in C_{opt}$ that can get mapped to the same $v' \in C_{gre}$ by the assumed clawfree nature of the graph. Why?
- It cannot happen that $k+1$ elements in C_{opt} get mapped to the same element in C_{gre} since this would either imply the existence of a $k+1$ claw or that the $k+1$ vertices in C_{opt} are not independent, neither of which can be true.
- For the weighted case, we observe that if $\nu \in C_{opt}$ is mapped to $v' \in C_{gre}$, then $w(\nu) \leq w(v')$. Why?
- This implies that the weight of vertices in C_{opt} is at most k times the weight of vertices in C_{gre} conclusing the proof.

Local Search for MIS on $k+1$ clawfree graphs

- Can we do better than a k-approximation for (W)MIS on $k+1$ clawfree graphs and hence for the (weighted) k set packing problem?

Local Search for MIS on $k + 1$ clawfree graphs

- Can we do better than a k-approximation for (W)MIS on $k + 1$ clawfree graphs and hence for the (weighted) k set packing problem?

- Khanna et al and Yu and Goldschmidt showed that we can get $\frac{k+1}{2}$ approximation for unweighted MIS on $k + 1$ clawfree graphs using a "2-improvement" (i.e. essentially a Hamming distance 3) local search.

  - We can achieve $\frac{k}{2} + \epsilon$ approximation using t-improvement local search.

- In the weighted case, it can be shown that the locality gap for the oblivious distance t local search algorithm is $(k - 1 + \frac{1}{t})$-approximation for weighted k-set packing.

- We will, however, soon see that we can improve upon the approximation ratio by either using a greedy initial solution and oblivious local search, or by using a non-oblivious local search.

Oblivious local search for weighted MIS on $k+1$ claw free graphs

- A simple (and arguably the most natural) oblivious local search algorithm provides a k approximation for the weighted MIS problem on $k+1$ clawfree graphs.
- In fact, the algorithm below has locality ratio k; i.e. any local optimum is a k approximation to the global optimum. It is *not* a totality ratio.

Simple oblivious local search for weighted MIS

Input: $G = (V, E)$, with weight $w_i = w(v_i)$
Output: An Independent set $S \subseteq V$
$S := \varnothing$
While $\exists v_i \in$ such that $\quad w_i > w(N(v_i) \cap S)$
  % that is, if we add v_i to S and remove $N(v_i)$ in S
  % where $N(v_i) = \{v_j | (v_i, v_j) \in E\}$
 $S := S \cup \{v_i\} \setminus N(S_i)$
End While

Charging argument to establish locality ratio

Claim: The simple oblivious local search for weighted MIS is a k approximation for $k+1$ clawfree graphs. Moreover, the simple local search has local ratio $= k$.

- The proof is a charging argument where we charge the weight of an arbitrary solution (i.e. an OPT solution) to the weight of any local optimum S.
- More precisely, we will charge the weight of any $v \in OPT$ to the weights of vertices in S so that for any $x \in S$, at most at most k times the weight of x is charged to x.
- The charging argument is a little different than those we have encountered before.

Charging argument continued

- For any $v \in OPT \cap S$, we charge the weight of v to itself.
- We can therefore now assume $OPT \cap S = \varnothing$.
- Let $v \in OPT$ have neighbourhood $N(v) = \{v_1, \ldots, v_\ell\}$.
  $N(v) \cap S \neq \varnothing$ since otherwise the algorithm could have added v to S.
  Aside: We know $\ell \leq k$ since S is an independent set and G is clawfree, but we really don't need that fact.
- We charge the neighbours of v (in S) a proportional amount of v's weight. Namely, we charge $v_j \in Nbhd(v) \cap S$ the weight $w(v) \cdot \frac{w_j}{w_1 + \ldots w_\ell}$
- Because S is a local optimum, $w(v) \leq w_1 + \ldots + w_\ell$
- Since OPT is an independent set, any $v_j \in S$ can be charged by at most k vertices $v \in OPT$ since G is clawfree.