CSC 373: Algorithm Design and Analysis
Lecture 15

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February 13, 2013

Some materials are from Stephen Cook's IIT talk and Keven Wayne's slides.

Announcements and Outline

Announcements

- No lecture this Friday. There will be additional questions for the next assignment which is due March 1.
- Please hand back your test if you have not already done do. I need to record the grade.
- You must keep all graded work until the term is over just in case there is some inconsistency in the grades recorded and what you have.
- Please refer to the web page for my policy on regrading.
- Enjoy reading week.

Today's outline

- We continue complexity theory and NP-completeness
- More on polynomial time reductions
- Some NP-complete problems

NP-Complete Problems

- These are the hardest NP problems.
- A problem A is p-reducible to a problem B if an "oracle" (i.e. a subroutine) for B can be used to efficiently solve A.
- If A is p-reducible to B, then any efficient procedure for solving B can be turned into an efficient procedure for A.
- If A is p-reducible to B and B is in P, then A is in P.

Definition

A problem B is NP-complete if B is in NP and every problem A in NP is p-reducible to B.

Theorem

If A is NP*-complete and A is in* P*, then* $\mathrm{P} = \mathrm{NP}$.

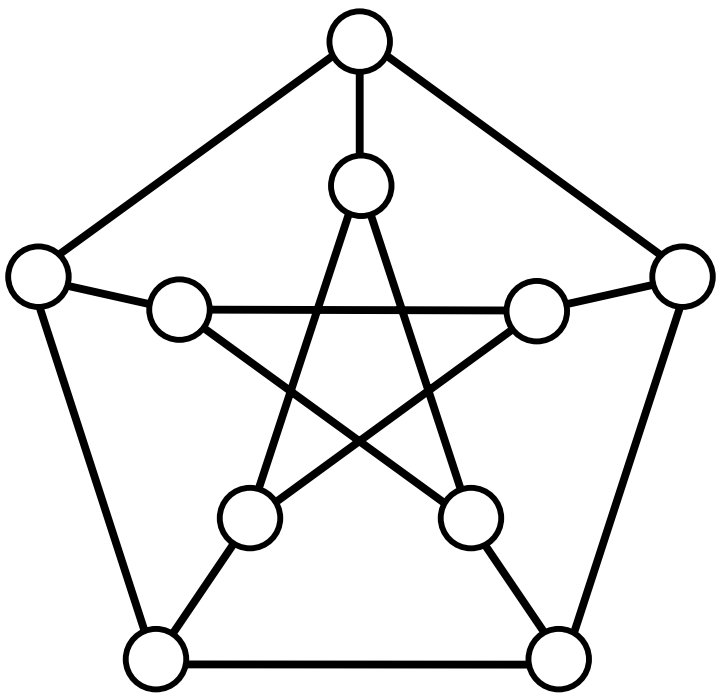
To show $\mathrm{P} = \mathrm{NP}$ you just need to find a fast (polynomial-time) algorithm for any one NP-complete problem!!!

Graph 3-Colorability

- A graph is a collection of nodes, with certain pairs of nodes connected by an edge.

Problem

Given a graph, determine whether each node can be coloured red, blue, or green, so that the endpoints of each edge have different colours.



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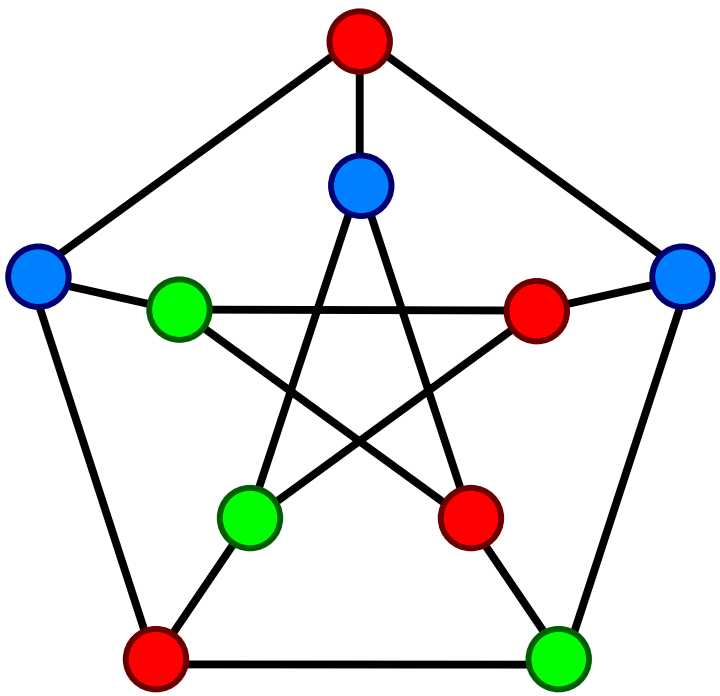


Some more remarks on graph coloring

- The natural graph coloring optimization problem is to color a graph with the fewest number of colors.
- We can phrase it as a search or decision problem by saying that the input is a pair (G, k) and then
  1. The search problem is to find a k-coloring of the graph G if one exists.
  2. The decision problem is to determine whether or not G has a k coloring.
  3. Clearly, solving the optimization problem solves the search problem which in turn solves the decision problem.
  4. Conversely, if we can efficiently solve the decision problem then we can efficiently solve the search and optimization problems.
- Formally it is the graph coloring decision problem which is NP-complete. More precisely, the decision problem for any fixed $k \geq 3$ is NP-complete. However, 2-Colorability is in P.
- But we will often abuse terminology and speak of the search problem or the optimization problem as being NP-complete.

Reducing Graph 3-Colourability to 3SAT

- We are given a graph G with nodes, say $V = \{v_1, v_2, \ldots, v_n\}$
- We are given a list of edges, say $(v_3, v_5), (v_2, v_6), (v_3, v_6), \ldots$
- We need to find a 3CNF formula F which is satisfiable if and only if G can be colored with 3 colors (say red, blue, green).
- We use Boolean VARIABLES
  $r_1, r_2, \ldots, r_n$ (r_i means node i is colored red)
  $b_1, b_2, \ldots, b_n$ (b_i means node i is colored blue)
  $g_1, g_2, \ldots, g_n$ (g_i means node i is colored green)

- Here are the CLAUSES for the formula F:
  - We need one clause for each node:
    $(r_1 \vee b_1 \vee g_1)$ (node 1 gets at least one color)
    $(r_2 \vee b_2 \vee g_2)$ (node 2 gets at least one color)
    ...
    $(r_n \vee b_n \vee g_n)$ (node n gets at least one color)
  - We need 3 clauses for each edge: For the edge (v_3, v_5) we need
    $(\overline{r_3} \vee \overline{r_5})$ (v_3 and v_5 not both red)
    $(\overline{b_3} \vee \overline{b_5})$ (v_3 and v_5 not both blue)
    $(\overline{g_3} \vee \overline{g_5})$ (v_3 and v_5 not both green)
- The size of the formula F is comparable to the size of the graph G.
- **Check:** G is 3-colorable if and only if F is satisfiable.

On the nature of this polynomial time reduction

- If we consider the previous reduction of 3-coloring to 3-SAT, it can be seen as a very simple type of reduction.
- Namely, given an input w to the 3-coloring problem, it is transformed (in polynomial time) to say $h(w)$ such that

$$w \in \{G \mid G \text{ can be 3-colored}\} \text{ iff}$$
$$h(w) \in \{F \mid F \text{ is a satisfiable 3CNF formula}\}.$$

- This was the same kind of polynomial time reduction that showed that bipartite matching is reducible to maximum flows.

Polynomial time transformations

- We say that a language L_1 is polynomial time transformable to L_2 if there exists a polynomial time function h such that

$$w \in L_1 \text{ iff } h(w) \in L_2.$$

- The function h is called a polynomial time transformation.

Polynomial time reductions and transformations

- In practice, when we are reducing one NP problem to another NP problem, it will be a polynomial time transformation.
- We will use the same notation $\leq_p$ to denote a polynomial time reduction and polynomial time transformation.
- As we have observed before if $L_1 \leq_p L_2$ and $L_2 \in \mathrm{P}$, then $L_1 \in \mathrm{P}$.
- The contrapositive says that if $L_1 \leq_p L_2$ and $L_1 \notin \mathrm{P}$, then $L_2 \notin \mathrm{P}$.

$\leq_p$ is transitive

- An important fact that we will use to prove NP completeness of problems is that polynomial time reductions are transitive.
- That is $L_1 \leq_p L_2$ and $L_2 \leq_p L_3$ implies $L_1 \leq_p L_3$.

- The proof for transformations is easy to see. For say that $L_1 \leq_p L_2$ via g and $L_2 \leq_p L_3$ via h, then $L_1 \leq_p L_3$ via $h \circ g$;
that is, $w \in L_1$ iff $h(g(w) \in L_3$.

Polynomial reductions/transformations continued

- One fact that holds for polynomial time transformation but is believed not to hold for polynomial time reductions is the following:

NP closed under polynomial time transformation

If $L_1 \leq_p L_2$ and $L_2 \in \text{NP}$ then $L_1 \in \text{NP}$.

- The closure of NP under polynomial time transformations is also easy to see. Namely,

Suppose

- $L_2 = \{w \mid \exists y, |y| \leq q(|w|) \text{ and } R(w, y)\}$ for some polynomial time relation R and polynomial q, and
- $L_1 \leq_p L_2$ via h.

Then

$L_1 = \{x \mid \exists y', |y'| \leq q(|h(x)| \text{ and } R'(x, y')\}$ where $R'(x, y') = R(h(x), y')$

Polynomial reductions/transformations continued

- On the other hand we do not believe that NP is closed under general polynomial time reductions.
- Specifically, for general polynomial time transformations we have $\bar{L} \leq_p L$. Here $\bar{L} = \{w | w \notin L\}$ is the language complement of L.
- We do not believe that NP is closed under language complementation.
- For example, how would you provide a short verification that a propositional formula F is *not* satisfiable? Or how would you show that a graph G *cannot* be 3-coloured?
- While we will use polynomial time transformations between decision problems/languages we need to use the general polynomial reductions to say reduce a search or optimization problem to a decision problem.

So how do we show that a problem is NP complete?

- Showing that a language (i.e. decision problem) L is NP complete involves establishing two facts:

1. L is in NP
2. Showing that L is NP-hard; that is showing

$$L' \leq_p L \text{ for every } L' \in \text{NP}$$

- Usually establishing 1 is relatively easy and is done directly in terms of the definition of $L \in$ NP.
  - That is, one shows how to verify membership in L by exhibiting an appropriate certificate. (It could also be established by a polynomial time transformation to a known $L \in$ NP.)
- Establishing 2, i.e. NP-hardness of L, is usually done by reducing some known NP complete problem L' to L.

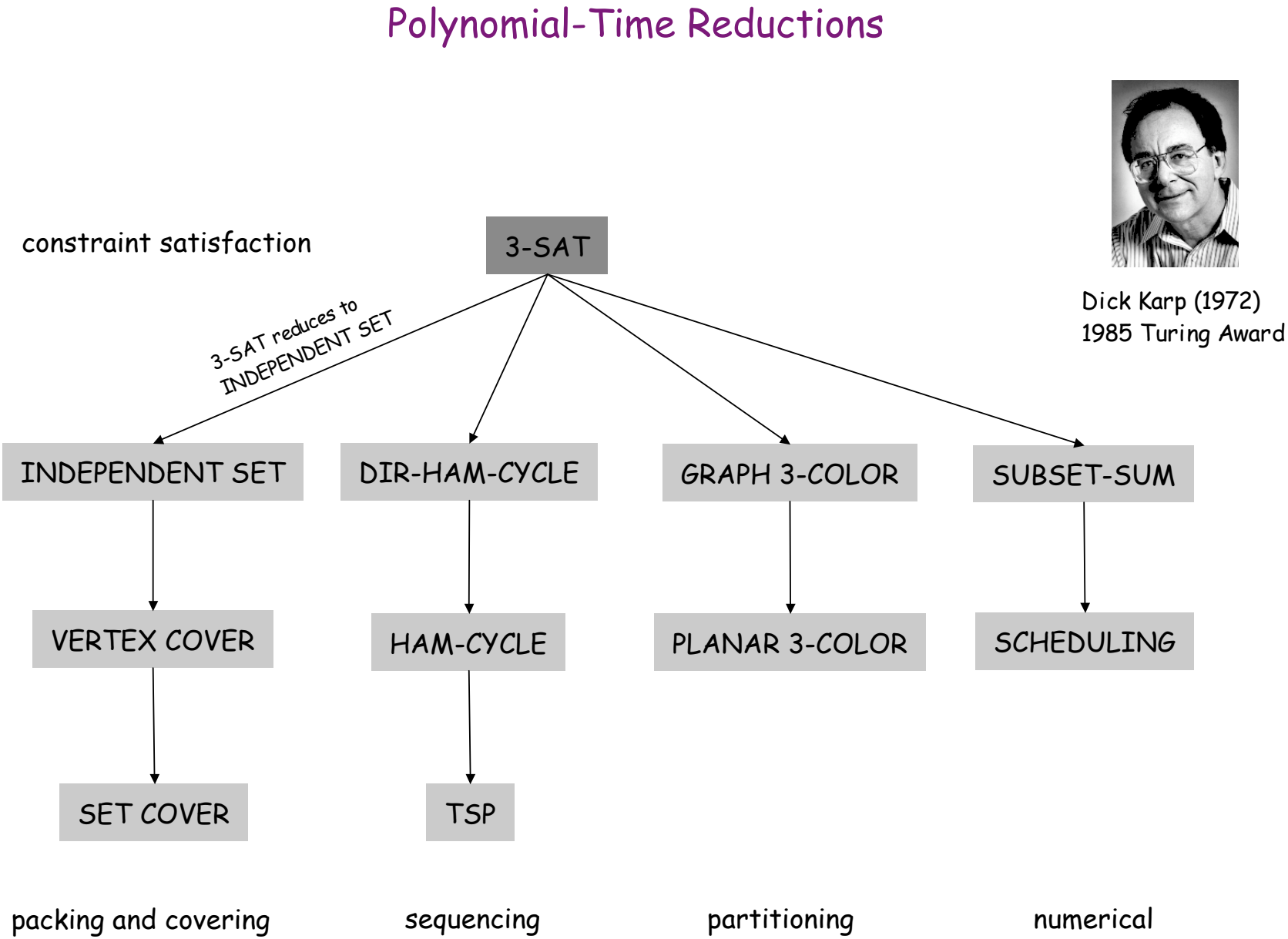
But how do we show that there are any NP complete problems?

How do we get started?

- Once we have established that there exists at least one NP complete problem then we can use polynomial time reductions and transitivity to establish that many other NP problems are NP hard.
- Following Cook's original result, we will show that SAT (and even $3SAT$) is NP complete "from first principles".

- It is easy to see that SAT is in NP.
- We will (later) show that SAT is NP hard by showing how to encode an arbitrary polynomial time (Turing) computation by a CNF formula.

A tree of reductions/transformations



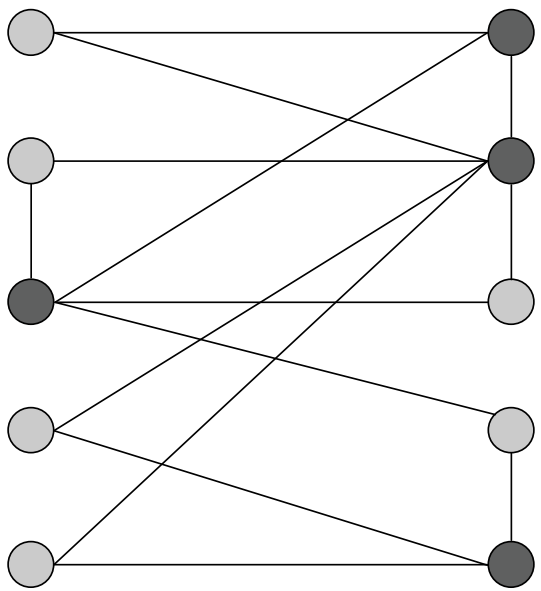
A little history of NP-completenes

- In his original 1971 seminal paper, Cook was interested in theorem proving. Stephen Cook won the Turing award in 1982
- Cook used the general notion of polynomial time reducibility which is called polynomial time Turing reducibility and sometimes called Cook reducibility.
- Cook established the NP completeness of 3SAT as well as a problem that includes CLIQUE $= \{(G, k) | G$ has a k clique $\}$.
- Independently, in the (former) Soviet Union, Leonid Levin proved an analogous result for SAT (and other problems) as a search problem.
- Following Cook's paper, Karp exhibited over 20 prominent problems that were also NP-complete.
- Karp showed that polynomial time transformations (sometimes called polynomial many to one reductions or Karp reductions) were sufficient to establish the NP completness of these problems.

Independent Set is NP complete

The independent set problem

- Given a graph $G = (V, E)$ and an integer k.
- Is there a subset of vertices $S \subseteq V$ such that $|S| \geq k$, and for each edge at most one of its endpoints is in S?



- Question: Is there an independent set of size 6?

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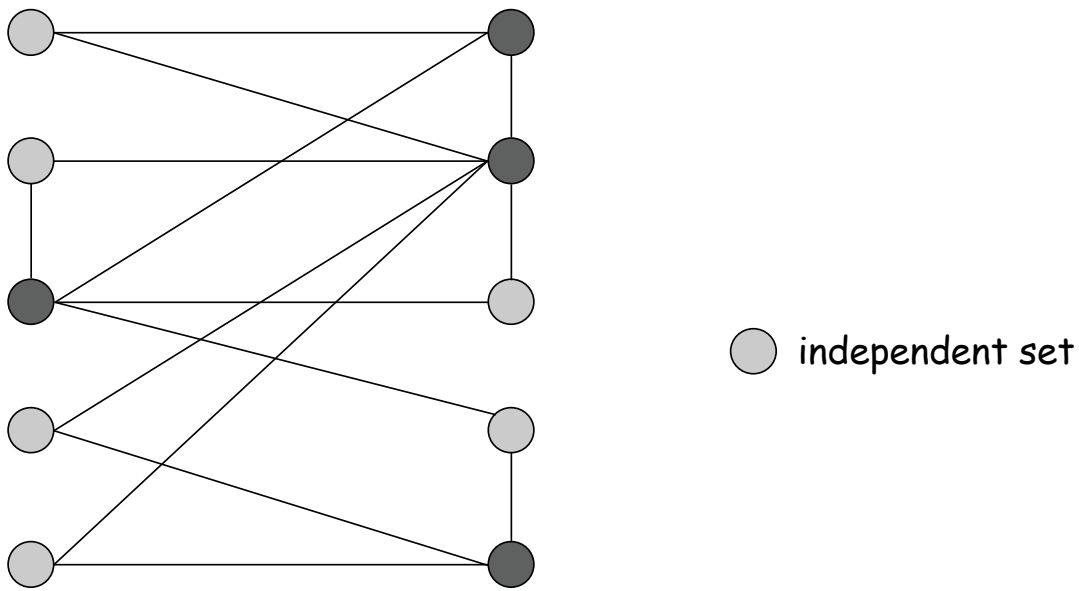

- Question: Is there an independent set of size 6? Yes.

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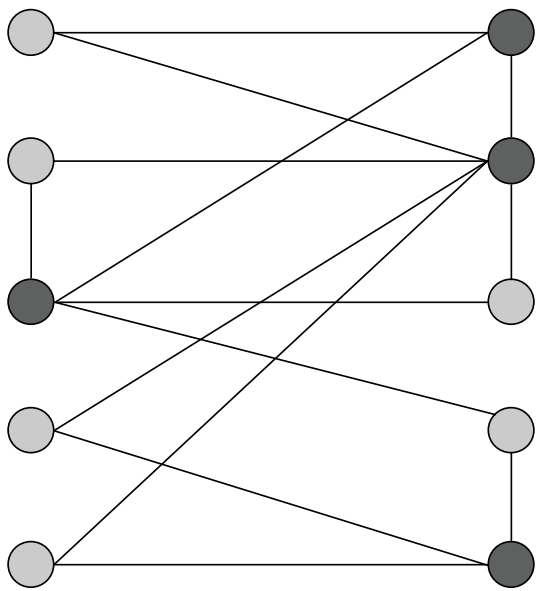
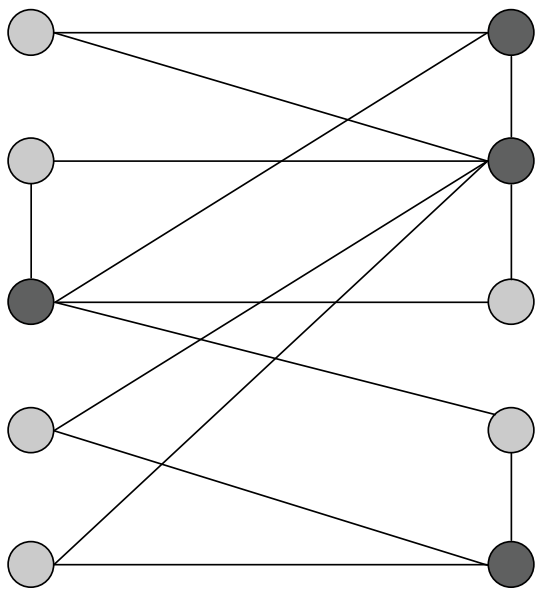


- Question: Is there an independent set of size 6? Yes.
- Question: Is there an independent set of size 7?

Independent Set is NP complete

The independent set problem

- Given a graph $G = (V, E)$ and an integer k.
- Is there a subset of vertices $S \subseteq V$ such that $|S| \geq k$, and for each edge at most one of its endpoints is in S?



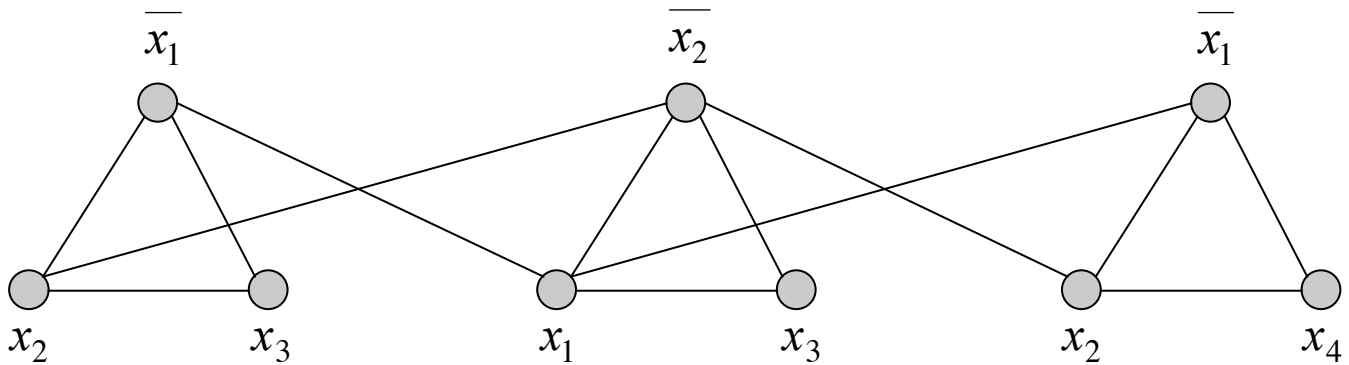
- Question: Is there an independent set of size 6? Yes.
- Question: Is there an independent set of size 7? No.

3SAT reduces to Independent Set

Claim

3SAT $\leq_p$ Independent Set

- Given an instance F of 3SAT, we construct an instance (G, k) of Independent Set that has an independent set of size k iff F is satisfiable.
- G contains 3 vertices for each clause; i.e. one for each literal.
- Connect 3 literals in a clause in a triangle.
- Connect literal to each of its negations.



$$\Phi = \left(\overline{x_1} \vee x_2 \vee x_3\right) \wedge \left(x_1 \vee \overline{x_2} \vee x_3\right) \wedge \left(\overline{x_1} \vee x_2 \vee x_4\right)$$

Subset Sum

The independent set problem

- Given a set of integers $S = \{w_1, w_2, \ldots, w_n\}$ and an integer W.
- Is there a subset $S' \subseteq S$ that adds up to exactly W?

Example

- Given $S = \{1, 4, 16, 64, 256, 1040, 1041, 1093, 1284, 1344\}$ and $W = 3754$.
- Question: Do we have a solution?
- Answer: Yes. $1 + 16 + 64 + 256 + 1040 + 1093 + 1284 = 3754$.

3SAT reduces to Subset Sum

Claim

3SAT $\leq_p$ Subset Sum

- Given an instance F of 3SAT, we construct an instance of Subset Sum that has solution iff F is satisfiable.

$$C_1 = \bar{x} \vee y \vee z$$
$$C_2 = x \vee \bar{y} \vee z$$
$$C_3 = \bar{x} \vee \bar{y} \vee \bar{z}$$

	x	y	z	C_1	C_2	C_3
x	1	0	0	0	1	0
¬ x	1	0	0	1	0	1
y	0	1	0	1	0	0
¬ y	0	1	0	0	1	1
z	0	0	1	1	1	0
¬ z	0	0	1	0	0	1
dummies to get clause columns to sum to 4	0	0	0	1	0	0
	0	0	0	2	0	0
	0	0	0	0	1	0
	0	0	0	0	2	0
	0	0	0	0	0	1
	0	0	0	0	0	2
W	1	1	1	4	4	4

Clay Math Institute Millenium Problems: $1,000,000 each

1. Birch and Swinnerton-Dyer Conjecture
2. Hodge Conjecture
3. Navier-Stokes Equations
4. P = NP?
5. Poincaré Conjecture **(Solved)**[1]
6. Riemann Hypothesis
7. Yang-Mills Theory

[1] Solved by Grigori Perelman 2003: Prize unclaimed

How important is the P vs NP question

Lance Fortnow has an article on P and NP in the September 2009 Communications of the ACM, in which he says

> *"The P versus NP problem has gone from an interesting problem related to logic to perhaps the most fundamental and important mathematical question of our time, whose importance only grows as computers become more powerful and widespread."*

Claim: It is worth well over the $1,000,000