CSC 373: Algorithm Design and Analysis
Lecture 12

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Lecture 12: Announcements and Outline

Announcements

- Term test 1 in tutorials. Need to use only two rooms due to sickness and conference. We will be using BA 2145 and BA 2155.

Today's outline

- Diiferent ways to choose an augmenting path so as to ensure polynomial time termination
- Immediate applications of the max-flow problem

A consequence of the max-flow min-cut theorem

Corollary

If all capacities are integral (or rational), then any implementation of the Ford-Fulkerson algorithm will terminate with an optimal integral max flow.

Rational capacities

Why does the claim about integral capacities imply the same for rational capacities?

The runtime of Ford-Fulkerson

Observation

Each augmenting path has residual capacity at least one.

- The max-flow min-cut theorem along with the above observation ensures that with integral capacities, Ford-Fulkerson must always terminate and the number of iterations is at most:

$$C = \text{the sum of edge capacities leaving } s.$$

- Hence complexity is $O(m + nC)$.

Notes

- There are bad ways to choose augmenting paths such that with irrational capacities, the Ford-Fulkerson algorithm will not terminate.
- However, even with integral capacities, there are bad ways to choose augmenting paths so that the Ford-Fulkerson algorithm does not terminate in polynomial time.

Bad example for naive Ford-Fulkerson

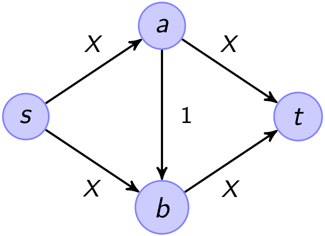

Figure: The numbers denote the capacities of the edges.

- The max-flow is clearly $2X$.
- A naive Ford-Fulkerson algorithm could alternate between
  - pushing a 1 unit flow along the augmenting path $s \to a \to b \to t$
  - pushing a 1 unit flow along the augmenting path $s \to b \to a \to t$
- This leads to a runtime of $\Omega(X)$, which is exponential if say X is given in binary.

Some ways to achieve polynomial time

- Choose an augmenting path having shortest distance: This is the Edmonds-Karp method and can be found in CLRS. It has running time $O(nm^2)$, where $n = |V|$ and $m = |E|$.
- There is a "weakly polynomial time" algorithm in KT
  - Here the number of arithmetic operations depends on the length of the integral capacities.
  - It follows that always choosing the largest capacity augmenting path is at least weakly polynomial time.
- There is a history of max flow algorithms leading to a recent $O(mn)$ time algorithm (see http://tinyurl.com/bczkdfz).
- The method I like to present (although not the fastest) is Dinitz's algorithm which has runtime $O(n^2 m)$.
  - A shortest augmenting-path method based on the concept of a blocking flow in the leveled graph.
  - Has some additional advantages beyond the somewhat better running time of Edmonds-Karp.

Dinitz's algorithm

Definition

- Define $level(u)$ = length of shortest path from s to u in G_f.
- Let the "leveled graph" w.r.t the residual graph G_f be the graph $L_f = (\hat{V}, \hat{E})$ where
  - $\hat{V} = \{v \mid v \text{ reachable from } s\}$
  - $(u, v) \in \hat{E}$ if and only if $level(v) = level(u) + 1$ in G_f.
- A blocking flow $\tilde{f}$ is a flow such that every s-t path in L_f has a saturated edge (i.e. an edge e such that $\tilde{f}(e) = c_f(e)$).

Dinitz's algorithm

Initialize $f(e) = 0$ for all $e \in E$.
while t is reachable from s in G_f **do**
    Construct L_f from G_f
    Find a blocking flow $\tilde{f}$ w.r.t. L_f and set $f := f + \tilde{f}$
end while
% There's no more augmenting path

Proof Sketch

Claims

1. The algorithm halts in at most $n - 2$ iterations.
2. A blocking flow in the leveled graph can be found in time $O(mn)$.

Proof.

Let f be a flow. Let f' be the updated flow after one iteration of Dinitz's algorithm, and let level' be the updated level w.r.t. the graph $G_{f'}$.

1. The proof of this claim rests on two facts:
   - For every node $v \in L_{f'}$, $\mathit{level}'(v) \geq \mathit{level}(v)$ since every edge in $L_{f'}$ is either an edge in G_f or the reverse of an edge in L_f.
   - Since f' was a blocking flow, $\mathit{level}'(t) > \mathit{level}(t)$.

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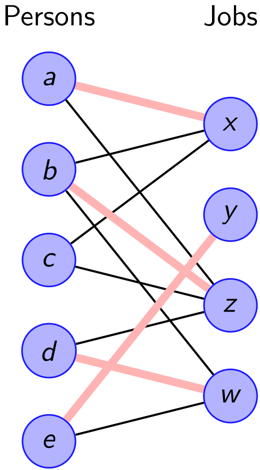
   - Since f' was a blocking flow, $\mathit{level}'(t) > \mathit{level}(t)$.
2. The leveled graph can be computed in $O(m)$. And using depth first search we can compute a blocking path in time $O(mn)$.

□

An application of max-flow: the maximum bipartite matching problem

The maximum bipartite matching problem

- Given a bipartite graph $G = (V, E)$ where
  - $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \varnothing$
  - $E \subseteq V_1 \times V_2$
- **Goal:** Find a maximum size matching.



- We do not know any efficient DP or greedy optimal algorithm for solving this problem.
- But we can efficiently reduce this problem to the max-flow problem.

The reduction

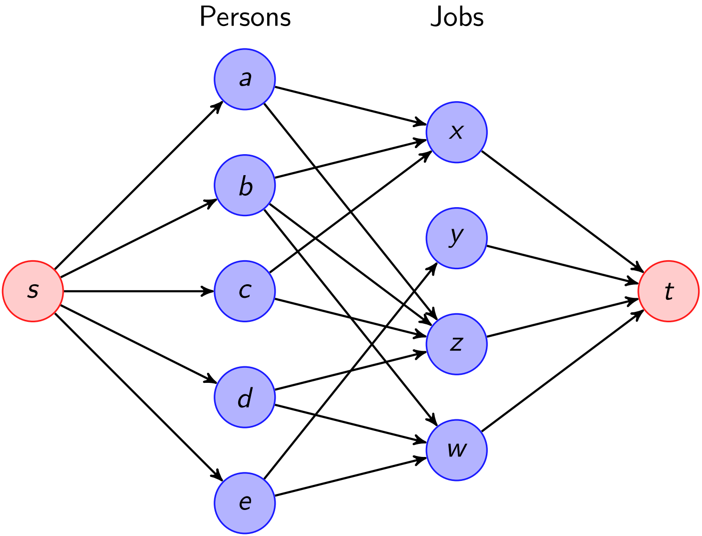


Figure: Assign every edge of the network flow a capacity 1.

The reduction preserves solutions

Claims

1. Every matching M in G gives rise to an integral flow f_M in the newly constructed network flow F_G with $val(f_M) = |M|$
2. Conversely every integral flow f in F_G gives rise to a matching M_f in G with $|M_f| = val(f)$.

Time complexity for bipartite matching using this reduction.

- $O(mn)$ using any Ford Fulkerson algorithm since the max flow is at most n and all capacities are integral.
- Dinitz's algorithm can be used to obtain a runtime $O(m\sqrt{n})$.

A few more comments on this reduction

- When we get to our next big topic (NP completeness), we will be focusing on decision problems and as a decision problem we have $|M| \geq k$ iff $val(f_M) \geq k$.
- The reduction we are using is very efficient (linear time in the representation of the graph) and it is a special type of polynomial time reduction which we will call a polynomial time transformation.

Alternating and augmenting paths in graphs

There is a graph theoretic concept of an augmenting path relative to a matching (in an arbitrary graph).

- An alternating path π relative to a matching M is one whose edges alternate between edges in M and edges not in M.
- An augmenting path is an alternating path that starts and ends with nodes not in M.

- The reduction provides a 1-1 correspondence between augmenting paths in the bipartite G w.r.t. M_f and augmenting paths in G_{f_M}.

Can this be extended to maximum weighted bipartite matching?

- In the weighted bipartite matching problem we are given an edge weighted bipartite graph $G = (V, E)$ with $V = V_1 \cup V_2$ and say integral weights $w : E \rightarrow \mathbf{N}$
- **Goal:** compute a matching M so as to maximize $\sum_{e \in M} w(e)$.
- A more or less obvious idea now is to form a flow network with new distinguished source and terminal nodes s and t.
- We would then set the capacity of the directed edge (x, y) to be $c(x, y) = w(x, y)$ for all $(x, y) \in E$.
- For edges leaving s and entering t we set

$$c(s, x) = \max_y \{w(x, y) : (x, y) \in E\}$$

$$c(y, t) = \max_x \{w(x, y) : (x, y) \in E\}$$

- Why doesn't this work?

Disjoint paths: Another similar application of max flow

- A natural problem of interest in fault tolerant networks is to ensure that there are sufficiently many edge disjoint paths.
- Given a directed graph $G = (V, E)$ with a distinguished source node s and terminal node t.
- **Goal:** compute the maximum number of edge disjoint paths from s to t.
- Similar to the bipartite matching transformation, we view G as a flow network $\mathcal{F}_G$ by setting the capacity of all edges equal to 1.
- Once again, because of integrality and unit capacities, we can argue that there are k edge disjoint paths in G iff $\mathcal{F}_G$ has (integral) flow k.
- And hence we can deduce Menger's theorem which states that the maximum number of edge-disjoint s-t paths in a directed graph is equal to the minimum number of edges in an s-t cut.
- The same theorem holds for undirected graphs.

The {0,1} metric labeling problem

- We now wish to consider one more application of max flow-min cut. Namely, we will consider the {0,1} metric labeling problem as discussed in §12.6 and §7.10 of Kleinberg and Tardos.
- This is in fact a special case of a more general metric labeling problem defined as follows:
  - The input is an edge weighted graph $G = (V, E)$, a set of labels $L = \{a_1, \ldots, a_r\}$ in a metric space with distance metric d, and functions $w : E \rightarrow \Re^{\geq 0}$ and $c : V \times L \rightarrow \Re^{\geq 0}$.
  - $c(u, a_j)$ is the cost of giving label a_j to node u.
  - **Goal:** find a labeling λ of the nodes $\lambda : V \rightarrow L$ so as to minimize $\sum_u c(u, \lambda(u)) + \sum_{(u,v)\in E} w(u, v) \cdot d(\lambda(u), \lambda(v))$
- For example, the nodes might represent documents, the labels are topics and the edges are links between documents weighted by the importance of the link.
- When there are 3 or more labels the problem is NP-hard even for the case of the $\{0, 1\}$ metric d for which $d(a_i, a_j) = 1$ for $a_i \neq a_j$ (and $d(a, a) = 0$ by the definition of a metric).

The labeling problem with 2 labels

- When there are only 2 labels, the only metric is the $\{0, 1\}$ metric.
- While the labeling problem is NP-hard for 3 or more labels, it is solvable in polynomial time for 2 labels by reducing the problem to the min cut problem.
- There is a also a 2-approximation algorithm for the $\{0, 1\}$ metric and 3 or more labels by another reduction to min cut. (And there are other non constant appoximation algorithms for arbitrary metrics.)
- Informally, the idea is that we can construct a flow network such that the nodes on the side of the sourse node s will correspond to say nodes labeled a and the node on the side of the terminal node t will correspond to the nodes labeled b.
- We will place capacities between the source s and other nodes to reflect the cost of a mislabel and similarly for the termnal t.
- The min cut will then correspond to a min cost labeling.