1. (20 points)

   A graph $G = (V, E)$ is called $(k + 1)$-clawfree if the neighbourhood $N(u)$ of every vertex u has at most k independent vertices. Here $N(u) = \{v \in V | (u, v) \in E\}$. Let $G = (V, E)$ be a $k + 1$-clawfree graph with positive node weights; say w_u is the weight of node u and let $w(S) = \sum_{u \in S} w_u$ be the sum of weights in $S \subseteq V$. Consider the following local search algorithm for approximately finding a maximum weight independent subset in a $k + 1$ clawfree graph $G = (V, E)$.

   $S := \emptyset$
   While $\exists u \in V - S$ such that $w_u > w(N(u) \cap S)$
   $\quad S := S \cup \{u\} - N(S)$
   End While

   Show the locality gap (and hence the approximation ratio) for this local-search algorithm is at most k for every weighted $(k + 1)$-clawfree graph.

   Solution

   Let OPT be an arbitrary (say optimal solution) and let S be a local optimum. Since S is a local optimum we know that for every $u \in V - S, w_u \leq w(N(u) \cap S)$. We are going to construct a mapping $h : w(OPT) \rightarrow S$ in such a way that for every $v \in S, w(OPT) \cap S \leq k \cdot w_v$. For clarity of explanation, lets assume that all node weights are positive integers. (Note: I am just making this assumption to simplify the discussion.) If we wish then to be more precise we can give names to each unit in weight $w(OPT)$; say for node $u \in OPT$, think of its weights being comprised of weight units $w_{u,1}, \ldots, w_{u,w_u}$. The mapping $h(u)$ maps the weight units of $u \in OPT$ to the vertices in $N(u) \cap S$ never mapping more than w_v units to any node $v \in (N(u) \cap S)$. Suppose $N(u) \cap S = \{v_1, \ldots v_r\}$. The idea is simply to map the first w_{v_1} units to v_1, the next w_{v_2} units to v_2, etc. (until all the weight units in u have been exhausted). The local optimality insures that the maaping $h(u)$ does not map more than w_{v_j} units on any $v_j \in N(u) \cap S$. (One could give an algorithm for computing h but I am hoping that this is clear.) To complete the proof, observe that the $(k + 1)$-clawfree property insures that for any v there are at most k nodes u such that $v \in N(u)$ so that the total weight mapped onto any $v \in S$ by h is at most $k \cdot w_v$ and hence all the units comprising $w(OPT)$ are mapped onto S, never mapping more than $k \cdot OPT$ units onto S.

2. (20 points) Consider the following *3 to 2 frequency set cover problem* : We are given a collection of sets $\mathcal{S} = \{S_1, \ldots, S_m\}$ for $S_i \subseteq U$ with the property that every $u \in U$ occurs in exactly three different sets S_i in $\mathcal{S}$. There is also a cost function $c : \mathcal{S} \rightarrow \Re^{\geq 0}$ and we let c_i denote the cost of set S_i. A feasible solution is a sub-collection $\mathcal{S}' \subseteq \mathcal{S}$ such that every $u \in U$ occurs in at least two different sets S_i in $\mathcal{S}'$. The goal is to find a feasible solution $\mathcal{S}'$ so as to minimize the cost $c(\mathcal{S}') = \sum_{S_i \in \mathcal{S}'} c_i$.

(a) Formulate the *3 to 2 frequency set cover problem* as a $\{0,1\}$ IP
Solution

IP minimnize $\sum c_i x_i$

subj. to $\sum_{i:u\in S_i} x_i \geq 2$ for each $u \in U$
$x_i\{0,1\}$ for $i = 1,\ldots,m$

Here the intended meaning is that $x_i = 1$ if and only S_i is in the cover. Notice that the inequalities guarantee that each u will be contained in at least two sets S_I in the cover.

(b) Show how to use LP relaxation + rounding to obtain an 2-approximation algorithm. Explain why your rounded solution is a feasible solution to the IP and why it provides a 2-approximation.

Solution

LP We replace $x_i \in \{0,1\}$ by $0 \leq x_i \leq 1$.

NOTE: Here as we will see we need the $x_i \leq 1$ condition.

Let $\{x_i^*\}$ be an LP optimum solution. We deterministically round by setting $\bar{x}_i = 1$ iff $x_i^* \geq 1/2$. We claim that the $\{\bar{x}_i\}$ are an IP solution (i.e. a solution to the given problem). Consider an arbitrary u and lets say $u \in S_1, S_2, S_3$. Then we have the inequality $x_1 + x_2 + x_3 \geq 2$. Suppose at most one of these three sets is in the cover determined by the $\{\bar{x}_i\}$. This means that at most one x_1^*, x_2^*, x_3^* have value $\geq 1/2$. Then since each $x_i \leq 1$, the sum $x_1^* + x_2^* + x_3^*$ is less than $1 + 1/2 + 1/2 < 2$ which is a contradiction. Since each variable is at most doubled by the rounding it follows that the rounded solution is at most twice the LP-OPT and hence at most twice the integral OPT.

3. (20 points)

Consider the following call routing problem. There is an n node bi-directional ring network $G = (V,E)$ upon which calls must be routed. That is $V = \{0,1,\ldots,n-1\}$ and $E = \{(i, i{+}1 \; mod \; n)\} \cup \{(i, i{-}1 \; mod \; n)\}$ and calls c_j are pairs (s_j, f_j) originating at node s_j and terminating at node f_j. Each call can be routed in a clockwise or counter-clockwise direction. The load L_e on any directed edge e is the maximum number of calls routed on this edge. The goal is to minimize $max_{e\in E} L_e$.

(a) Formulate this problem as an IP.
Hint: Consider variables x_j and y_j that indicate the direction of call c_j. (You can also use just one indicator variable to represent the direction but I think it might be easier to think in terms of two such variables.)

Solution

Let S_j be the set of clockwise edges that a call C_j would be routed on if the call is routed clockwise. Similarly let T_j be the set of counter-clockwise edges that a call C_j would be routed on if the call is routed counter-clockwise. Note that the sets S_j and T_j are disjoint sets determined by the call C_j.

IP: minimize L

subj. to $\sum_{j:e \in S_j} x_j \leq L$ for each edge e

$\sum_{j:e \in T_j} y_j \leq L$ for each edge e

$x_j + y_j = 1$ for each call C_j

$x_j, y_j \in \{0, 1\}$ for each call C_j

The intended meaning is that x_j (resp y_j) is 1 if and only if C_j is routed clockwise (resp. counter-clockwise).

The conditions $x_j + y_j = 1$ and $x_j, y_j \in \{0, 1\}$ for each call C_j insure that each call is routed in exactly one direction. The other inequalities insure that no more than L calls are routed on any edge.

(b) Using an LP relaxation of this problem, show how to derive a 2-approximation algorithm.

Solution

We simply round up iff the LP-OPT varianble is at least 1/2. The proof that this is an IP solution is immediate. The proof thst the rounded solution is at most twice the LP-OPT is obso immediate.