CSC373F Fall 2011

Charging argument for EFT

- We will use a (somewhat inelegant) charging argument to prove that the EFT greedy algorithm for the JISP problem is a 2-approximation algorithm; that is, $||OPT| \leq 2|EFT|$. This is bascially the same as the argument showing that EFT is optimal for ISP. In fact, we are going to prove something stronger, namely we will define a function $h : OPT \rightarrow EFT$ such that h is 2-1; that is, for every $I_k \in EFT$, there are at most two intervals, say I_j and I_ℓ such that $h(I_j) = h(I_\ell) = I_k$.

- Without loss of generality we will restrict attention to intervals I in $OPT - EFT$ for if I is in EFT as well as OPT, then $h(I) = I$ and there cannot be another I' in OPT that intersects I.

- The function is defined by mapping any I_j in OPT to the left-most interval (say I_k) in EFT that is not compatible; that is, I_k intersects I_j or has the same job class number (i.e. $c_j = c_k$). Since EFT is greedy, there must be such an I_k or else I_j would be taken by EFT and then $h(I_j) = I_j$.

- Let I_k be any interval in EFT (and by our assumption assume $I_k \notin OPT$. It remains to show that h is 2-1. That is, there are at most two intervals in OPT that can be mapped to I_k. It is clear that there can be at most one interval (say I_j) such that $c_j = c_k$. Whenever that is the case, we will map I_j to I_k. So it only remains to show that there can be at most one $I_j \in OPT$ that is charged to I_k because of interval intersection. So suppose I_j intersects I_k and $h(I_j) = I_k$. We will need to consider some cases:

  1. Case 1 $f_j < f_k$. Since we are assuming that $I_j \notin EFT$, there has to be a reason that EFT did not take I_j before it took I_k. That is, there must have been another $I_\ell \in EFT$ with $\ell < k$ that is not compatible with I_j. But then h would map I_j to that I_ℓ.
  2. Case 2: $f_k \leq f_j$
     (a) Case 2a: If $s_j \leq s_k$, then interval I_j includes I_k and therefore no other interval I in OPT can intersect I_k.
     (b) Case 2b: $s_k < s_j < f_k \leq f_j$. If there is another $I_r \in OPT$ that intersects I_k, it must be that $f_r < s_j$ since otherwise $f_j < s_r$ and hence I_r cannot intersect I_k. Hence $f_r < f_k$. But then as in case 1, the reason that I_r was not taken by EFT is because it is incompatible with some interval I_ℓ with $\ell < k$ and hence I_r would have been mapped to that I_ℓ.

CLAIM: We will see (on the next page) how to "clean up" this proof by abstracting away the geometry. But sometimes (as in the m machine interval scheduling problem) we need the geometry.

- We say that a graph $G = (V, E)$ is a chordal graph if it has a perfect elimination ordering of its vertices; namely an ordering $v_1, v_2, \ldots, v_n$ such that the "inductive neighbourood" $Nbhd(v_i) \cap \{v_{i+1}, \ldots, v_n\}$ is a clique (equivalently has at most on3 independent vertex).

- We observe that as interval graph G = (V,E) induced by intervals $I_1, \ldots, I_n$ is chordal by ordering the vertices representing intervals so that $f_1 \leq f_2 \leq \ldots \leq f_n$.

- We generalize the EFT algorithm to a greedy algorithm for ISP and JISP (I guess we could call it PEO-greedy) for chordal graphs. Namely we use the PEO as the ordering of vertices and then accept greedily.

- We prove that this PEO-greedy is optimal for the ISP problem by a charging argument; namely we show that there is a 1-1 function $h : OPT \rightarrow$ PEO-greedy. Again without loss of generality we need only consider vertices in OPT but not in PEO-greedy. Dfine $h(v_j) = v_k$ where $k = argmin_k\{(v_j, v_k) \in E$ and $v_k \in$ PEO-greedy. As before h is a well defined function and it only remains to show that h is 1-1.

- Let $v_k \in$ PEO-greedy. Suppose that there are two (or more) vertices v_j and v_ℓ in OPT (not in greedy) such that $h(v_j) = h(v_\ell) = v_k$. We have two cases:

    1. Case 1: At least one of these intervals (say j) is such that $j < k$. i Then there must be a reason PEO-greedy didnt take j since we are assuming $v_j \notin$ PEO-greedy. Namely, there is a vertex $v_r \in$ PEO-greedy with $r < j < k$ such that $(v_r, v_j) \in E$. But then v_j should have been mapped to v_r.
    2. Case 2: $k < j < \ell$. But then $Nbhd(v_k) \cap \{v_{k+1}, \ldots, v_n\}$ has at least two independent vertices contradicting the assumption that $v_1, \ldots, v_n$ is a PEO.

- That completes the proof. To extend this proof to show that PEO-greedy is a 2-approximation for JISP, we note that the intersection graph for this problem leads to a class of graphs where there is an ordering of vertices such that $Nbhd(v_i) \cap \{v_{i+1}, \ldots, v_n\}$ has at most two independent vertices. And then the same proof shows that the same mapping h is 2-1.

- An equally easy proof shows that using the reverse of a PEO ordering and then greedily colouring becomes an optimal greedy algorithm for colouring a chordal graph.