Charging argument for EFT

- We will use a (somewhat inelegant) charging argument to prove that the EFT greedy algorithm for the JISP problem is a 2-approximation algorithm; that is, $|OPT| \leq 2|EFT|$. This is basically the same as the argument showing that EFT is optimal for ISP. In fact, we are going to prove something stronger, namely we will define a function $h : OPT \rightarrow EFT$ such that $h$ is 2-1; that is, for every $I_k \in EFT$, there are at most two intervals, say $I_j$ and $I_\ell$ such that $h(I_j) = h(I_\ell) = I_k$.

- Without loss of generality we will restrict attention to intervals $I$ in $OPT - EFT$ for if $I$ is in $EFT$ as well as $OPT$, then $h(I) = I$ and there cannot be another $I'$ in $OPT$ that intersects $I$.

- The function is defined by mapping any $I_j$ in $OPT$ to the left-most interval (say $I_k$) in $EFT$ that is not compatible; that is, $I_k$ intersects $I_j$ or has the same job class number (i.e. $c_j = c_k$). Since EFT is greedy, there must be such an $I_k$ or else $I_j$ would be taken by $EFT$ and then $h(I_j) = I_j$.

- Let $I_k$ be any interval in $EFT$ (and by our assumption assume $I_k \notin OPT$. It remains to show that $h$ is 2-1. That is, there are at most two intervals in $OPT$ that can be mapped to $I_k$. It is clear that there can be at most one interval (say $I_j$) such that $c_j = c_k$. Whenever that is the case, we will map $I_j$ to $I_k$. So it only remains to show that there can be at most one $I_j \in OPT$ that is charged to $I_k$ because of interval intersection. So suppose $I_j$ intersects $I_k$ and $h(I_j) = I_k$. We will need to consider some cases:

1. Case 1 $f_j < f_k$. Since we are assuming that $I_j \notin EFT$, there has to be a reason that EFT did not take $I_j$ before it took $I_k$. That is, there must have been another $I_\ell \in EFT$ with $\ell < k$ that is not compatible with $I_j$. But then $h$ would map $I_j$ to that $I_\ell$.

2. Case 2: $f_k \leq f_j$
   (a) Case 2a: If $s_j \leq s_k$, then interval $I_j$ includes $I_k$ and therefore no other interval $I$ in $OPT$ can intersect $I_k$.
   (b) Case 2b: $s_k < s_j < f_k \leq f_j$. If there is another $I_\ell \in OPT$ that intersects $I_k$, it must be that $f_\ell < s_j$ since otherwise $f_j < s_\ell$ and hence $I_\ell$ cannot intersect $I_k$. Hence $f_\ell < f_k$. But then as in case 1, the reason that $I_\ell$ was not taken by $EFT$ is because it is incompatible with some interval $I_\ell$ with $\ell < k$ and hence $I_\ell$ would have been mapped to that $I_\ell$.

CLAIM: We will see (on the next page) how to “clean up” this proof by abstracting away the geometry. But sometimes (as in the $m$ machine interval scheduling problem) we need the geometry.
- We say that a graph $G = (V, E)$ is a chordal graph if it has a perfect elimination ordering of its vertices; namely an ordering $v_1, v_2, \ldots, v_n$ such that the “inductive neighbourhood” $\text{Nbhd}(v_i) \cap \{v_{i+1}, \ldots, v_n\}$ is a clique (equivalently has at most one independent vertex).

- We observe that as interval graph $G = (V, E)$ induced by intervals $I_1, \ldots, I_n$ is chordal by ordering the vertices representing intervals so that $f_1 \leq f_2 \leq \ldots \leq f_n$.

- We generalize the EFT algorithm to a greedy algorithm for ISP and JISP (I guess we could call it PEO-greedy) for chordal graphs. Namely we use the PEO as the ordering of vertices and then accept greedily.

- We prove that this PEO-greedy is optimal for the ISP problem by a charging argument; namely we show that there is a 1-1 function $h: \text{OPT} \rightarrow \text{PEO-greedy}$. Again without loss of generality we need only consider vertices in OPT but not in PEO-greedy. Define $h(v_j) = v_k$ where $k = \text{argmin}_k \{(v_j, v_k) \in E \text{ and } v_k \in \text{PEO-greedy}\}$. As before $h$ is a well defined function and it only remains to show that $h$ is 1-1.

- Let $v_k \in \text{PEO-greedy}$. Suppose that there are two (or more) vertices $v_j$ and $v_\ell$ in OPT (not in greedy) such that $h(v_j) = h(v_\ell) = v_k$. We have two cases:

  1. Case 1: At least one of these intervals (say j) is such that $j < k$. Then there must be a reason PEO-greedy didn’t take $v_j$ since we are assuming $v_j \notin \text{PEO-greedy}$. Namely, there is a vertex $v_r \in \text{PEO-greedy}$ with $r < j < k$ such that $(v_r, v_j) \in E$. But then $v_j$ should have been mapped to $v_r$.

  2. Case 2: $k < j < \ell$. But then $\text{Nbhd}(v_k) \cap \{v_{k+1}, \ldots, v_n\}$ has at least two independent vertices contradicting the assumption that $v_1, \ldots, v_n$ is a PEO.

- That completes the proof. To extend this proof to show that PEO-greedy is a 2-approximation for JISP, we note that the intersection graph for this problem leads to a class of graphs where there is an ordering of vertices such that $\text{Nbhd}(v_i) \cap \{v_{i+1}, \ldots, v_n\}$ has at most two independent vertices. And then the same proof shows that the same mapping $h$ is 2-1.

- An equally easy proof shows that using the reverse of a PEO ordering and then greedily colouring becomes an optimal greedy algorithm for colouring a chordal graph.