CSC 373 Lecture 25

Announcements:

Graded term test 2 and assignments will be ready Wednesday and can be discussed in tutorial on Thursday.

So far one request for TA office hour.

Today

- Start IP/LP rounding.
  - Vertex cover
  - Makespan on unrelated machines.

IP and LP relaxations

- We now begin our last algorithmic paradigm, namely integer programming IP and linear programming LP. Mainly we will be discussing the LP relaxation of IPs and rounding such LPs to obtain IP solutions.
- We will start with some examples and then briefly discuss some LP theory. Mainly we are treating LP as a black box.

Complexity status

- While there are problems which are directly represented by LPs (eg max flow), I will focus on NP hard problems which are (in most cases) naturally represented by IPs. Indeed solving IPs is an NP-hard problem although there are many heuristics and special cases that are solvable in practice and sometimes in theory. LPs are efficiently solvable, both in practice and theoretically (polynomial time but not known to be strongly poly time) although they do not tend to be as efficient as simpler combinatorial methods.

LPs in standard form

- For a minimization problem, the *standard form* is to minimize sum $c_i * x_i$ subject to m linear constraints $A*\mathbf{x} \geq \mathbf{b}$; $\mathbf{x} \geq \mathbf{0}$.
- For maximization, the standard form is to maximize sum $c_i * x_i$ subject to m linear constraints $A*\mathbf{x} \leq \mathbf{b}$; $\mathbf{x} \geq \mathbf{0}$.
- “Duality” uses standard form. Minimization (resp. maximization) problem for which all a_{ij} and b_i are non negative are called covering (resp. packing problems).
- There is another basic form called slack form where slack variables are used to turn all inequalities into equalities and that is a convenient form for the Simplex method.

Overview of the IP formulation for vertex cover and its LP relaxation

- Weighted vertex cover as a {0,1} IP, its LP relaxation and “naïve” rounding .
- Why this is a 2-approximation.
- The integrality gap of this IP/LP relaxation
- Adding some additional inequalities (say corresponding to odd cycles) does not help.
- In general there will be many IP formulations for a given problem. An integrality gap pertains to one (or a class of) IP and LP relaxations.

An IP for weighted vertex cover.

In standard form, we want to minimize sum $w_i * x_i$ subj to $x_i + x_j >= 1$ and x_i in {0,1}.

The intended meaning is that $x_i = 1$ iff vertex v_i is in the cover.

The LP relaxation is to relax the integrality condition to $0 <= x_i$. In this problem it follows that an optimal LP solution also satisfies $x_i <=1$.

Rounding the LP optimal

- Suppose ***x**** is an LP optimum. We can apply a "naïve" rounding (naive in the sense that the rounding ignores the input) to the fractional solution by setting *x'_i = 1* iff *x*_i >= ½*.
- Clearly ***x'*** is an integral solution to the IP and hence *V' = {v_i: x'_i = 1}* is a vertex cover.
- Claim: the weight of the cover *V'* is at most twice the weight of an optimal cover. Because the LP is a relaxation it allows more solutions and hence *LP-OPT <= IP-OPT*. Thus *w(V') <= 2 LP-OPT <= 2 IP-OPT*.

The integrality gap

- Given a naive rounding, this IP/LP relaxation is at best a (*2-1/n*) approximation due to the following considerations:
- For the complete (unweighted) graph on *n* nodes, the optimal IP value is *n-1* where as the LP optimum value is *n/2*. The ratio *(n-1)/(n/2)* is the integrality gap.
- For the *n* node cycle, the optimum IP solution is (the ceiling of) *n/2*.
- There are many ways to "tighten" the IP formulation (the easiest being to add inequalities for triangles or all odd length cycles(but the integrality gap remains.

Set cover and *f*-frequency set cover

- We are given a collection of (possibly weighted) sets $C = \{S_1, \ldots S_n\}$ over a universe U. The set cover problem is to find a minimal size (weight) subcollection C' that covers all the elements in U.
- Set cover generalizes vertex cover and turns out to be hard to approximate (given well believed assumptions about NP) better than H_m where $m = |U|$ is the size of the Universe. There is a natural greedy algorithm that will achieve an approximation of H_d where $d = \max_i |S_i|$.
- *f*-frequency set cover and vertex cover as 2-frequency set cover problem with $U = E$ and sets $S_i = \{e \mid e$ adjacent to vertex $v_i\}$.