CSC373F **Problem Set 4** **Fall, 2005**

Due: Friday, Dec 2 , beginning of lecture
Note change of due date by one week. Questions 2 and 5 have now become bonus questions.

NOTE: Each problem set only counts 5% of your mark, but it is important to do your own work (but see below). Similar questions will appear on the first term test which will cover material relating to both assignment 1 and assignment 2. You may consult with others concerning the general approach for solving problems on assignments, but you must write up all solutions entirely on your own. the University's Code of Behavior. You will receive 1/5 points for any question/subquestion for which you say "I do not know how to answer this question". You will receive .5/5 points if you just leave the question blank.

1. Show how to formulate the following problems as $\{0,1\}$ integer programming (IP) problems

   (a) The MaxSat problem where the input is a $F = C_1 \wedge C_2 \ldots C_r$ where each clause is a disjunction of literals.
   Hint: If there are r clauses and n propositional variables occuring in F, then it suffices to have $r + n$ variables occuring in the IP.

   Solution: Introduce a $\{0,1\}$ variable u_i for each propositional variable x_i and a $\{0,1\}$ variable z_j for each clause C_j. The objective function is to maximize $\sum_j w_j z_j$ subect to the inequalities:
   $\sum_{x_i \in P(C_j)} u_i + \sum_{x_i \in N(C_j)} (1 - u_i) \geq z_j$ for each clause C_j
   where $P(C_j)$ (respectively, $N(C_j)$) are the variables that occur positively (respectively, negatively) in C_j. For example, in the clause $C = x_1 \vee x_2 \vee \overline{x_3}$, x_1 and x_2 are in $P(C)$ and x_3 is in $N(C)$. Note that, if $z_j = 0$ so that we are not taking that clause, the inequality for C_j is trivially satisfied. If $z_j = 1$ then at least one literal must be satisifed. The inequality insures that this is the case where we are using $u_i = 0$ (respectively, $u_i = 1$) to mean that x_i is being set false (respectively, true).

   (b)

2.

3.

4. Consider the following *dominating set problem*: We are given a collection of sets $\mathcal{T} = \{T_1, \ldots, T_m\}$ with $T_i \subseteq U$ for some universe U. There is also a cost function $c : U \rightarrow \Re^{\geq 0}$ and we let c_u denote the cost of element $u \in U$. A feasible solution is a subset $S \subseteq U$ such that $S \cap T_i \neq \emptyset$ for all i. The goal is to find a feasible subset S so as to minimize the cost $c(S) = \sum_{u \in S} c_u$.

(a) Formulate the dominating set problem as a $\{0,1\}$ IP

Solution: We introduce a $\{0,1\}$ variable x_u for each $u \in U$ with the intended meaning that $x_u = 1$ iff we take u in the solution S. So the objective function is to minimize $\sum c_u \cdot x_u$ subject to the inequalities:
$\sum_{u \in T_i} x_u \geq 1 \quad$ for each T_i.
Clearly if all inequalities are satisfied then each T_i intersects at least one $u \in S = \{u | x_u = 1\}$.

(b) Show how to use LP relaxation + rounding to obtain an d-approximation algorithm in the case that $|T_i| \leq d$ for every set T_i in the collection $\mathcal{T}$.

Solution: If $|T_i| \leq d$ for every set T_i, then in each inequality there are at most d variables x_u. Let $(x_1^*, \ldots, x_m^*)$ be a fractionally optimal solution to the LP relaxation obtained by replacing $x_u \in \{0,1\}$ by $x_u \geq 0$. Then round each x_u^* to $\overline{x_u} = 1$ iff $x_u^* \geq \frac{1}{d}$. Since each inequality has at most d variables, then for any inequality to be satisfied in the LP, for at least one variable x_u^* we must have $x_u^* \geq \frac{1}{d}$.

5.