Name ______________________ Student No. ______________________

AIDS ALLOWED: One page (two sides) of handwritten notes
Answer ALL questions on test paper. Use backs of sheets for additional space.
Total Marks: 100

REMINDER: You get 20% of any question or subquestion if you state "I do not know how to answer this question". You get 10% of any question which you just leave blank.

1. The following TRUE/FALSE questions all concern a flow network $\mathcal{F} = (G, s, t, c)$ where G is a directed graph, s is the source node, t is the terminal node and $c : E \to N$ is the capacity function. Let f be a max flow in $\mathcal{F}$. Answer each question providing a brief explanation for your answer. Note: an answer without an explanation will be treated the same as "I do not know how to answer this question". [30 points]

   (a) Suppose the capacity of *every* edge in G is decreased by 1 unit to form a new flow network $\mathcal{F}'$. Then for every $\mathcal{F}$ with max flow f, the value of the max flow in $\mathcal{F}'$ is always less than the value of the flow f.

   True. If f is a max flow then the value of the flow f = capcacity of some min cut C in the network $\mathcal{F}$. The capacity of this cut muts decrease in the new network $\mathcal{F}'$.

   (b) Suppose the capacity of one edge e^* is decreased by 1 unit to form a new flow network $\mathcal{F}'$. Then for every $\mathcal{F}$ with max flow f, the value of the max flow in $\mathcal{F}'$ is always less than the value of the flow f.

   False. The edge e^* need not be in any min cut and then reducing its capacity by one unit will not change the max flow. You can easily give an example.

(c) Suppose the capacity of every edge in G is multiplied by 2 to form a new flow network $\mathcal{F}'$. That is, $c'(e) = 2 \cdot c(e)$. Then for every $\mathcal{F}$ with max flow f, the value of the max flow in $\mathcal{F}'$ is always equal to twice the value of the flow f.

True. This can be thought of as just a "rescaling". More precisely, it is clear that the value of all min cuts is now twice as large.

2. Given a flow network $\mathcal{F}$ as in question 1, give a LP (Linear Progamming) formulation for the max flow optimization problem. [20 points]

   An LP for max flow can be optained by introducing a variable f_e for every edge e in the network. If s is the source node, then the LP is to maximize the objective function $\sum_{e:e=(s,v)} f_e$ subject to the conditions defining an allowable flow f. Namely, for each $v \neq s, t$, there is a flow conservation equality which can be stated as two inequalities, anbd for each edge e, we also have the capacity constraint that $f_e \leq c_e$. Finally, we have $f_e \geq 0$ for each edge e.

3. Consider the following variant of the f- frequency set cover problem. As before, there is a universe $U = \{u_1, \ldots, u_n\}$ and a collection of subsets $\mathcal{S} = \{S_1, \ldots, S_m\}$; each set S_i has a positive cost $c(S_i)$ and every universe element occurs in at most f of the sets in $\mathcal{S}$. . The goal is now to cover every element in the universe at least 2 times by some sub-collection $\mathcal{C} \subseteq \mathcal{S}$ so as to minimize the cost of this subcollection. Give an IP formulation for this problem. Give an LP relaxation to the IP and show how to utilize this LP formulation so as to achieve a $2f$-approximation to the optimization problem. [30 points]

   Everything here is the same as in assignment 3, except now for each element $u \in U$, the constraint $\sum_{S:u \in S} c_S \geq 1$ is replaced by $\sum_{S:u \in S} c_S \geq 2$. Given an optimal solution x for the relaxed LP, we round every variable x_S to x^*_S by setting $x^*_S = 1$ if x^*
$x_S \geq 1/2f$ and 0 otherwise. We now argue as in the assignment that x^* is a solution to the IP and clearly it is within a factor of $2f$ of the optimal LP solution.

4. Consider the min capacity $s - t$ cut problem. That is, given a graph $G = (V, E)$ with capacity function $c : E \to N$, find a cut (S, T) with $s \in S$ and $t \in T$ so as to minimize $\sum_{e:e=(u,v),u\in S,v\in T} c(e)$. Suppose we modify the contraction algorithm so that in each iteration it chooses (uniformly at random) to contract an edge $e = (u, v)$ subject to the condition that e will not contract the supernode containing s and the supernode containing t. Show that, for any polynomial fundtion p, this modified contraction algorithm will NOT find a min capacity $s - t$ cut with probability at least $1/p(n)$.
Hint: Consider the example showing that there can be exponentailly many min $s-t$ cuts in a graph. Use edge capacities so that the modified algorithm is unlikely to find a min capacity $s - t$ cut. [20 points]

Consider the graph with nodes $\{s, u_1, u_2, \ldots, u_n, t\}$ and edges (s, u_i) with capacity 1, (u_i, t) with capacity 2, for each u_i. The modified contraction algorithm can only contract a node u_i with either the supernode containing s or the supernode containing t. Since it has only probability $2/3$ to contatct any u_i witht eh supernode containing s, the probability will only be $(2/3)^n$ that it makes the correct choice for every u_i.