CSC 364S Notes **University of Toronto, Spring, 2003**

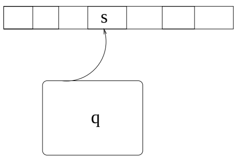
Turing Machines and Polynomial Time

So far, we have discussed a number of problems for which we were able to come up with polynomial time algorithms, and some for which we were not able to find a polynomial time algorithm. We never *rigorously* defined, however, what we meant by an "algorithm", or what it means for an algorithm to run in polynomial time. This lack of a formal definition doesn't matter much as long as we are coming up with descriptions of procedures that people accept as sufficiently efficient algorithms. However, once we want to start arguing that efficient algorithms (for a particular problem) *don't* exist, we have to be completely rigorous about what we mean by an efficient algorithm.

In 1936, Alan Turing gave the first definition of an algorithm, in the form of (what we call today) a Turing machine. He wanted a definition that was simple, clear, and sufficiently general. Although Turing was only interested in defining the notion of "algorithm", his model is also good for defining the notion of "polynomial-time" algorithm. (We tend to use "polynomial time" as a rigorous surrogate for "efficient"; this is very useful, even though not all polynomial-time algorithms are truly efficient.)

First, we introduce some notation.
Let Σ be a finite alphabet of symbols, By Σ^* we mean the set of all finite strings (including the empty string) consisting of elements of Σ. By a *language* over Σ we mean a subset $L \subseteq \Sigma^*$. For a string $x \in \Sigma^*$, we denote the length of x by $|x|$.

Turing machines



A Turing machine consists of a two-way infinite tape and a finite state control. The tape is divided up into squares, each of which holds a symbol from a finite tape alphabet that includes the blank symbol $\not b$.

The machine has a read/write head that is connected to the control, and that that scans squares on the tape. Depending on the state of the control and symbol scanned, it makes a move, consisting of

- printing a symbol
- moving the head left or right one square
- assuming a new state

The tape will initially be completely blank, except for an input string over the finite input alphabet Σ; the head will initially be pointing to the leftmost symbol of the input.

A Turing machine M is specified by giving the following:

- Σ (a finite input alphabet).
- Γ (a finite tape alphabet). $\Sigma \subseteq \Gamma$. $\not{b} \in \Gamma - \Sigma$.
- Q (a finite set of states). There are 3 special states:
  - q_0 (the initial state)
  - q_{accept} (the state in which M halts and accepts)
  - q_{reject} (the state in which M halts and rejects)
- $\delta : (Q - \{q_{accept}, q_{reject}\}) \times \Gamma \to Q \times \Gamma \times \{L, R\}$ (the transition function)
  (If $\delta(q, s) = (q', s', h)$, this means that if q is the current state and s is the symbol being scanned, then q' is the new state, s' is the symbol printed, and h is either L or R, corresponding to a left or right move by one square.

The Turing machine M works as follows on input string $x \in \Sigma^*$.
Initially x appears on the tape, surrounded on both sides by infinitely many blanks, and with the head pointing to the leftmost symbol of x. (If x is the empty string, then the tape is completely blank and the head is pointing to some square.) The control is initially in state q_0.
M moves according to the transition function δ.
M may run forever, but if it halts, then it halts either in state q_{accept} or q_{reject}. (For the moment, we will only be interested in whether M accepts or rejects; later, we will explain what it means for M to output a string.)

Definition 1. *M* accepts *a string $x \in \Sigma^*$ if M with input x eventually halts in state q_{accept}. We write $\mathcal{L}(M) = \{x \in \Sigma^* | M \text{ accepts } x\}$, and we refer to $\mathcal{L}(M)$ as the language accepted by M.*

(Notice that we are abusing notation, since we refer both to a Turing machine accepting a string, and to a Turing machine accepting a language, namely the set of strings it accepts.)

Example 1. *PAL = the set of even length palindromes = $\{yy^r | y \in \{0,1\}^*$, where y^r means y spelled backwards.*

We will design a Turing machine M that accepts the language PAL$\subseteq \{0,1\}^*$. M will have input alphabet $\Sigma = \{0,1\}$, and tape alphabet $\Gamma = \{0, 1, \not{b}\}$. (Usually it is convenient to let Γ have a number of extra symbols in it, but we don't need to for this simple example.) We will have state set $Q = \{q_0, q_1, q_2, q_3, q_4, q_5, q_{accept}, q_{reject}\}$.

If, in state q_0, M reads 0, then that symbol is replaced by a blank, and the machine enters state q_1; the role of q_1 is to go to the right until the first blank, remembering that the symbol 0 has most recently been erased; upon finding that blank, M enters state q_2 and goes left one square looking for symbol 0; if 0 is not there then we *reject*, but if 0 is there, then we erase it and enter state q_3 and go left until the first blank; we then goes right one square, enter state q_0, and continue as before.

If we read 1 in state q_0, then we operate in a manner similar to that above, using states q_4 and q_5 instead of q_1 and q_2.

If we read $\not b$ in state q_0, this means that all the characters of the input have been checked, and so we accept.

Formally, the transition function δ is as follows. (Note that when we accept or reject, it doesn't matter what we print or what direction we move in; we arbitrarily choose to print $\not b$ and move right in these cases.)

State q	*Symbol* s	*Action* $\delta(q, s)$
q_0	0	$(q_1, \not b, R)$
q_1	0	$(q_1, 0, R)$
q_1	1	$(q_1, 1, R)$
q_1	$\not b$	$(q_2, \not b, L)$
q_2	1	$(q_{reject}, 0, R)$
q_2	$\not b$	$(q_{reject}, 0, R)$
q_2	0	$(q_3, \not b, L)$
q_3	0	$(q_3, 0, L)$
q_3	1	$(q_3, 1, L)$
q_3	$\not b$	$(q_0, \not b, R)$
q_0	1	$(q_4, \not b, R)$
q_4	0	$(q_4, 0, R)$
q_4	1	$(q_4, 1, R)$
q_4	$\not b$	$(q_5, \not b, L)$
q_5	0	$(q_{reject}, 0, R)$
q_5	$\not b$	$(q_{reject}, 0, R)$
q_5	1	$(q_3, \not b, L)$
q_0	$\not b$	$(q_{accept}, 0, R)$

We now define the notion of worst case time complexity of Turing machines.

Let M be a Turing machine over input alphabet Σ. For each $x \in \Sigma^*$, let $t_M(x)$ be the number of steps required by M to *halt* (i.e., terminate in one of the two final states) on input x. (Each step is an execution of one instruction of the machine, and we define $t_M(x) = \infty$ if M never halts on input x.)

Definition 2 (Worst case time complexity of M). *The worst case time complexity of M is the function $T_M : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ defined by*

$$T_M(n) = \max\{t_M(x) \mid x \in \Sigma^*, |x| = n\}.$$

Let us return to the Turing machine M from our palindrome example, and try to estimate $T_M(n)$. Calculating $T_M(n)$ exactly is difficult, but we can find reasonable upper and lower

bounds for it. To compute an upper bound, consider an arbitrary input x of length n; M will make less than or equal to $n+1$ passes over x, each taking less than or equal to $n+1$ steps, so $t_M(x) \leq (n+1)^2$, so $T_M(n) \in O(n^2)$. For a lower bound, consider a palindrome of length n (where n may or may not be even). M will go to the right for $n+1$ steps, then left for n steps, etc., for a total of $(n+1)+(n)+\cdots+1 \in \Omega(n^2)$ steps.
So $T_M(n)$ is in $\Theta(n^2)$.

(Notice that M performs much faster on some strings than on others. For example, if $x = 10000000$, then $t_M(x) = 10$. In general, if the first and the last symbols of x differ, then $t_M(x) = n+2$.)

Now we are ready to formally define the class **P** of polynomial-time languages.

Definition 3 (Polynomial-time Turing Machine). *A Turing Machine M is* polynomial-time *if for some $c \in \mathbb{N}$, $T_M(n) \in O(n^c)$.*

Definition 4 (Formal definition of P).

$$\mathbf{P} = \{L \mid L = \mathcal{L}(M) \text{ for some polynomial-time Turing Machine } M\}.$$

It is clear that PAL$\in$**P**. We will discuss other languages in **P** below.

When we talk about Turing machines accepting a language, we implicitly are only concerned, for each input, with whether or not the input is accepted, that is, with whether we say "yes" or "no" on the input.

Often, however, we want our algorithm to output a *string*, so we need a convention whereby a Turing machine, when it halts, can be viewed as outputting a string. (Note that when we view a Turing machine as outputting a string, we will not care if the final state is accepting or not.) We will let the the output alphabet be the same as the input alphabet Σ. If and when machine M halts, we define the *output* to be the string $y \in \Sigma^*$ beginning at the head position, and continuing to the right up to but not including the first symbol not in Σ; since only a finite portion of the tape is non-blank and since $\not{b} \notin \Sigma$, such a symbol will exist.

We say that M computes $f : \Sigma^* \to \Sigma^*$ if for every $x \in \Sigma^*$, M computing on input x eventually halts with output $f(x)$. We can now formally define the set **FP** of functions computable in polynomial time.

Definition 5 (Formal definition of FP).

$$\mathbf{FP} = \{f : \Sigma^* \to \Sigma^* \mid M \textit{ computes } f \text{ for some polynomial-time Turing machine } M\}.$$

Before giving some more examples of languages in **P** and functions in **FP**, we should reflect on the reasonableness of our definitions of these classes. After all, Turing machines are pretty inconvenient to program, and the programs we get often seem inefficient. For this reason we will introduce a "suped-up" version of a Turing machine, called a "multi-tape Turing machine"; these machines are easier to program and are more efficient than normal Turing machines, but their polynomial-time power is just the same as normal Turing machines.

A multitape Turing machine M uses k tapes, for some constant $k \geq 1$. The first tape is the input-output tape, according to the same conventions as with a normal Turing machine; the other $k-1$ tapes are initially blank. The Finite State Control starts in state q_0 and has k heads, one reading/writing each tape. The move M makes depends on the state and on the k symbols being read; a move consists of writing on each tape, moving each head, and going to a new state. More formally the transition function is:

$$\delta : (Q - \{q_{accept}, q_{reject}\}) \times (\Gamma^k) \to Q \times (\Gamma \times \{L, R\})^k$$

As an example, consider (a description of) a 2-tape machine M for accepting PAL:
M begins by checking that the input x is of even length, and if not rejects. Then M copies x onto the second tape, and positions the head on tape 1 to the left of x and positions the head on tape 2 to the right of x. Then M moves one head to the right while moving the other to the left, checking that the symbols match; if they all match, M accepts, otherwise M rejects.
For this machine we have linear (worst-case) running time: $T_M(n) \in \Theta(n)$.

It turns out that we can always simulate a multitape Turing machine by a normal Turing machine, with the running time at worst squaring. The new Turing machine will be equivalent to the old one, in the sense that if the old one runs forever, so will the new one; if the old one halts and accepts (respectively rejects), then the new one will halt and accept (respectively reject) with the same output as the old one.

Theorem 1. *For every multitape Turing machine M there is an equivalent normal Turing machine M' such that $T_{M'}(n) \in O((T_M(n))^2)$.*

Proof Outline:
We want to simulate a k-tape machine M by a 1-tape machine M'. One way of doing this is as follows. A portion of the tape of M' will contain all the information about the squares of the tapes of M that have been visited. We will think of (this part of) the tape of M' as containing k (vertical) "tracks", where the ith track contains information about the ith tape of M, including the position of the head; a tape symbol of M' will therefore actually be a k-tuple of symbols. If on a square of M' the ith track has symbol a, this will mean that the corresponding square of tape i of M contains a; if on a square of M' the ith track has symbol a^*, this will mean that the corresponding square of tape i of M contains a and has the head pointing to it. If M has visited s tape squares so far, then M' will take time $O(s)$ to simulate the next step of M.

So on an input of length n, M' will have to simulate $T_M(n)$ steps, taking time $O(T_M(n))$ on each step, for a total time of $O((T_M(n))^2)$. $\square$

The Church-Turing Thesis, which will be discussed later, states that Turing machines properly formalize the notion of what we can compute. For the moment, we will state a version of this thesis, that states that we have properly formulated the notion of polynomial-time computation.

Polytime Thesis If a language can be accepted (or a function computed) by some sort of polynomial-time algorithm, according to some reasonable notion of polynomial-time algorithm, then that language or function can be computed by a polynomial-time Turing machine.

Although it is intuitively obvious that every polynomial-time Turing machine should be considered a polynomial-time algorithm, it is not at all clear that the converse of this, namely the Polytime Thesis, is true. There are a lot of objections one can think of. An obvious one is that maybe polynomial-time Random Access Machines (or RAMs) are more powerful than Turing machines. However, it is not hard to see how to simulate a polynomial-time RAM by a polynomial-time Turing machine, assuming some reasonable restrictions are placed on our definition of the RAM. (In particular, we insist that the word size is no more than a polynomial in the length of the input.)

A more potent objection is that it often appears to help if we are allowed to toss random coins during a computation; in this case, it is only required that we get the right answer with high probability. This objection can be countered by allowing our Turing machine to flip random coins as well. Another objection involves "quantum computers". We will not describe these here, except to say that there are certain tasks, such as factoring large integers (presented in binary), that quantum computers can perform in polynomial time, that *appear* to require exponential time on even probabilistic Turing machines. It is not clear, however, that it is possible to "physically construct" actual machines that approximate the behavior of quantum computers. It turns out that understanding what problems require exponential time is a very deep mathematical issue; it is also an extraordinarily deep issue in physics to understand to what extent a quantum computer can be built. The point of the Polytime Thesis, is that if we are able to construct machines that are much more powerful than Turing machines (at least allowing randomness), then our understanding of mathematics and of the universe will have been significantly altered.

One should note that that every language in **P** has a corresponding function in **FP**. Let $L \subseteq \Sigma^*$ be a language, where Σ contains both 0 and 1. Define the *characteristic function of* L, $C_L : \Sigma^* \to \Sigma^*$, by
$C_L(x) = 1$ if $x \in L$, and $C_L(x) = 0$ if $x \notin L$.
It is now easy to see that
$L \in \mathbf{P} \Leftrightarrow C_L \in \mathbf{FP}$.

Note that the definitions of the classes **P** and **FP** are sensitive to how the input is represented. For example, if the input numbers to the knapsack problem are expressed in unary notation, then the dynamic programming problem discussed earlier in the course solves this problem in polynomial time. However, if the numbers are expressed in binary notation (which is the default case), then this problem will turn out to be "**NP**-complete", which will imply that it probably cannot be solved in polynomial time.

Examples of Languages and Search Problems

Often we will describe a language in the following manner.

MSTD (Minimum Spanning Tree Decision Problem).
Instance:
$\langle G, B \rangle$ such that G is a connected, undirected graph with integer costs on the edges, and $B \in \mathbb{N}$ (with all integers represented in binary).

Acceptance Condition:
Accept if G has a spanning tree with cost $\leq B$.

The notation $\langle G, B \rangle$ means that the graph G and number B are represented in some standard form over a standard alphabet. By "Instance", we mean any string that is properly formed, in the manner described. Formally speaking, MSTD is just the set of $\langle G, B \rangle$ such that G is a connected, undirected graph with integer costs on the edges, and $B \in \mathbb{N}$, and G has a spanning tree with cost $\leq B$.

Claim MSTD $\in$ **P**.
A polynomial-time algorithm is: use Kruskal's algorithm to find a MST T; if T has cost $\leq B$, accept, otherwise reject.

The decision problem MSTD is related to the more natural "search problem" MST.

MST (Minimum Spanning Tree Search Problem).
Instance:
$\langle G \rangle$ such that G is a connected, undirected graph with integer costs on the edges (with all integers represented in binary).

Output:
Output a minimal cost spanning tree of G.

We will often describe search problems in this way. The goal, given an instance, is to find some appropriate output such as an optimal value or an optimal structure of some sort; often there are many different possible correct outputs.

We will say that a *search problem is computable in polynomial time*
if for some function $f \in$**FP**, $f(x)$ is a correct output for every instance x,
and $f(x)$ is the empty string for all non-instances x.
(The way we have chosen to deal with non-instances is very arbitrary. We could have instead used the convention that we regard non-instances as a particular, trivial instance.)

If we can solve a search problem in polynomial time, then we can solve the corresponding decision problem in polynomial time. Often the converse holds as well.

GKS (General Knapsack Search Problem).
Instance:
$\langle (w_1, g_1), \cdots, (w_m, g_m), W \rangle$ (with all integers represented in binary).

Output:
Output a feasible knapsack with highest possible profit.

GKD (General Knapsack Decision Problem).
Instance:
$\langle (w_1, g_1), \cdots, (w_m, g_m), W, B \rangle$ (with all integers represented in binary).

Acceptance Condition:
Accept if there is a feasible knapsack with profit $\geq B$.

SDPDS (Scheduling with Deadlines, Profits and Durations Search Problem).
Instance:
$\langle (d_1, g_1, t_1), \cdots, (d_m, g_m, t_m) \rangle$ (with all integers represented in binary).

Output:
Output a feasible schedule with highest possible profit.
(Note that the ordering of deadlines, profits and durations is different than in earlier notes: the $\{d_i\}$ are deadlines, the $\{g_i\}$ are profits, and the $\{t_i\}$ are durations.)

SDPDD (Scheduling with Deadlines, Profits and Durations Decision Problem).
Instance:
$\langle (d_1, g_1, t_1), \cdots, (d_m, g_m, t_m), B \rangle$ (with all integers represented in binary).

Acceptance Condition:
Accept if there is a feasible schedule with profit $\geq B$.

We will see in the next section that there are close relationships between these last four problems. Later on, we will see that because the languages are "**NP**-complete", there is very good reason to believe that none of these four problems have polynomial time algorithms.

Polynomial-Time Reducibilities

Let P_1 and P_2 be "problems", where by a "problem" we mean a either a language or a search problem. We will say that P_1 is polynomial-time reducible to P_2, and write $P_1 \xrightarrow{p} P_2$, if P_1 can be solved in polynomial time with the help of a P_2 solver.

Definition 6. *P_1 is polynomial-time reducible to P_2 ($P_1 \xrightarrow{p} P_2$) if there is a polynomial-time algorithm for P_1 which is allowed to access a solver for P_2, where the time taken by the P_2 solver is not counted.*

It will always be the case that the decision version of a problem is polynomial-time reducible to the search version.

Example 2 (GKD $\xrightarrow{p}$ GKS, and SDPDD $\xrightarrow{p}$ SDPDS).

To show, say, the first reduction, we want to solve GKD using GKS. Given an instance $x = \langle (w_1, g_1), \cdots, (w_m, g_m), W, B\rangle$ of GKD, just create $y = \langle (w_1, g_1), \cdots, (w_m, g_m), W\rangle$ and give y to a solver for GKS, getting S as an output. Next, we compute the profit g of S, and compare it to B. If $B \leq g$ then we accept, otherwise we reject. Clearly this only takes time polynomial in $|x|$ (ignoring the time taken by the GKS solver).

Intuitively it is clear that any problem that is polynomial-time reducible to a problem solvable in polynomial time, is itself solvable in polynomial time.

Theorem 2. *If $P_1 \xrightarrow{p} P_2$ and P_2 is solvable in polynomial time, then P_1 is solvable in polynomial time.*

Proof:
Say that M_2 solves P_2 in polynomial time. Let M be a Turing machine that solves P_1 in polynomial time, using a solver for P_2. (At this point, we should really give more of an explanation about our syntactic conventions whereby M gets access to a solver for P_2. One way to do this is to have a special tape where M writes inputs for P_2, and another special tape where the output from P_2 instantly appears.) Since M runs in time, say, $O(n^c)$ on inputs of length n, all of the inputs to the P_2 solver must be of length $O(n^c)$.

Assume that M_2 solves P_2 in time $O(n^d)$. We now describe a machine M_1 that solves P_1 in polynomial time. On an input of length n, M_1 will behave like M, but whenever M wants to solve a P_2 problem on an input x, M_1 will run M_2 on x; since $|x| \in O(n^c)$, this running of M_2 will take time $O(|x|^d)$, that is, time $O(n^{cd})$. M_1 will have to run M_2 at most $O(n^c)$ times, for a total running time of $O(n^{c+cd})$. $\square$

It will often be the case that the search version of a problem is polynomial-time reducible to the decision version.

Example 3 (GKS $\xrightarrow{p}$ GKD, and SDPDS $\xrightarrow{p}$ SDPDD).

We will only prove the first reduction; the second is similar and is left as an exercise.

Say that we are given an instance $x = \langle (w_1, g_1), \cdots, (w_m, g_m), W\rangle$ of GKS, where $|x| = n$. Our first goal is to find the value of the optimal profit, using a GKD solver. We are able to test, for any B we choose, whether or not it is possible to achieve profit at least B. Let GKD(B) be answer from GKD for the instance $x = \langle (w_1, g_1), \cdots, (w_m, g_m), W, B\rangle$. We *could* perform this test for $B = 1, B = 2, \cdots$ until we get the answer "no". However, since the profits $\{g_i\}$ are written in binary notation, the optimal profit can be as big as (about) 2^n, so this wouldn't run within polynomial time. Instead we will do what is in effect a binary search. Let C be the sum of all the g_i.

We will call GKD($\lceil \frac{C}{2} \rceil$).
If the answer is "yes", call GKD($\lceil \frac{3C}{4} \rceil$); if the answer is "no", call GKD($\lceil \frac{C}{4} \rceil$).
We continue reducing the size of the interval by a factor of 2 on each step, until a value B is obtained such that GKD(B) = "yes" and GKD($B+1$)="no". That B will be the value of the optimal profit.

Now, knowing the optimal profit B, we want to find a knapsack achieving profit B.
If GKD on $\langle (w_1, g_1), \cdots, (w_{m-1}, g_{m-1}), W, B \rangle$ returns "yes", then we can forget about item m, and find a solution to the $\langle (w_1, g_1), \cdots, (w_{m-1}, g_{m-1}), W \rangle$ knapsack problem with (optimal) profit B;
otherwise, we use item m, and find a solution to the $\langle (w_1, g_1), \cdots, (w_{m-1}, g_{m-1}), W - w_m \rangle$ knapsack problem with (optimal) profit $B - g_m$.
Continuing in this way, we find an optimal knapsack. It is easy to check that the running time is polynomial in n. $\square$

An important special case of $L_1 \xrightarrow{p} L_2$ is where L_1 and L_2 are languages, and the reduction is of a very special form: given input x, compute $f(x)$, and accept if and only if $f(x)$ is in L_2. We write $L_1 \leq_p L_2$, and say that L_1 is *polynomial-time transformable* to L_2.

Definition 7. *Let $L_1, L_2 \subseteq \Sigma^*$.*
Then $L_1 \leq_p L_2$ if for some $f : \Sigma^* \rightarrow \Sigma^*$,
*$f \in$ **FP** and for all $x \in \Sigma^*$, $x \in L_1 \Leftrightarrow f(x) \in L_2$.*

Easy Fact: If $L_1 \leq_p L_2$, then $L_1 \xrightarrow{p} L_2$, and therefore if $L_2 \in$ **P**, then $L_1 \in$ **P**.

We showed earlier how GKS can be regarded as a "special case" of $SDPDS$. We can formalize this assertion by stating:

Claim: GKS $\xrightarrow{p}$ SDPDS, and
GKD $\leq_p$ SDPDD.

To show GKD $\leq_p$ SDPDD, consider an input x for GKD. Assume x is an instance of GKD, $x = \langle (w_1, g_1), \cdots, (w_m, g_m), W, B \rangle$.
(It is easy to compute if x is an instance of GKD, and if it isn't, we just let $f(x)$ be some trivial string not in SDPDD.)
Then we let $f(x) = \langle (W, g_1, w_1), \cdots, (W, g_m, w_m), B \rangle$.
It is easy to see that f is computable in polynomial time,
and that for all $x \in \Sigma^*$, $x \in GKD \Leftrightarrow f(x)$ SDPDD.

A corollary of all this is that if Scheduling With Deadlines, Profits and Durations is polynomial-time computable, then the General Knapsack problem is polynomial-time computable. We will see later that the theory of **NP**-completeness implies that the converse is true as well.

It is important to realize that both $\xrightarrow{p}$ and $\leq_p$ are *transitive*. We will prove this for $\leq_p$.

Theorem 3. *Let $L_1, L_2, L_3 \subseteq \Sigma^*$. If $L_1 \leq_p L_2$ and $L_2 \leq_p L_3$, then $L_1 \leq_p L_3$.*

Proof:
Say that $L_1 \leq_p L_2$ via function $f_1 : \Sigma^* \to \Sigma^*$, where Turing machine M_1 computes f_1 in time $O(n^c)$ on inputs of length n; say that $L_2 \leq_p L_3$ via function $f_2 : \Sigma^* \to \Sigma^*$, where Turing machine M_2 computes f_2 in time $O(n^d)$ on inputs of length n.

Define $h : \Sigma^* \to \Sigma^*$ by $h(x) = f_2(f_1(x))$. Clearly for all $x \in \Sigma^*$,
$x \in L_1 \Leftrightarrow f_1(x) \in L_2 \Leftrightarrow f_2(f_1(x)) \in L_3 \leftrightarrow h(x) \in L_3$.

We can compute h by a machine M in polynomial time as follows: Let x be the input, $|x| = n$. M begins by computing $y = f_1(x)$ by running M_1 on x. This will take time $O(n^c)$, and y will have length $O(n^c)$. M then runs M_2 on y to compute $f_2(y) = h(x)$, in time $O(|y|^d) = O((n^c)^d) = O(n^{cd})$. The total time for M is $O(n^c) + O(n^{cd}) = O(n^{cd})$, a polynomial in n. $\square$