# CSC304: Algorithmic Game Theory and Mechanism Design Fall 2016

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# Lecture 7

- Announcements
  - Tutorial this Friday.
  - Tyrone Strangway will give a guest lecture Monday on equilibirum pricing.
  - Assignment due date is October 7.
  - Slides for Lecture 6 have been posted.
- Today's agenda
  - A quick comment on normal form vs extensive form games
  - Congestion games
  - Potential games
  - Braess paradox and the Price of Anarchy (POA)

# Normal vs extensive form games

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But it is important to note that almost all the concepts we have been discussing apply equally to both settings (with the exception being *subgame-perfect equilibrium*). More specifically: the following are relevant in both settings:

- Payoffs can be profits (to be maximized) or costs (to be minimized).
- Dominant strategies
- Mixed Strategies
- Pure and mixed Nash equilibrium
- Optimal and Pareto optimal strategies
- Bayesian settings
- Repeated games

# Congestion games

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Quoting wikipedia: Congestion games are a class of games consisting of players and resources where the payoff (or cost) to each player depends on the resources it uses and the number of players choosing the same resource. That is, and more generally, the payoff or cost depends on the congestion on each resource.

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Congestion games can be finite (i.e. finitely many players each having finitely many pure strategies) or infinite or even continuous (where there can be a continuum of players and/or strategies for a player). Where it is helpful, we can view a continuous congestion game as the limiting case as the number $n$ of players goes to infinity. Chapter 4 introduces an example (the road congestion game) of a finite congestion game and a continuous variant of this game is considered in Chapter 8.

# The road network congestion game

The finite road network congestion game consists of a finite number $k$ of players (i.e. drivers), each player $i$ needing to choose a path $P_i$ in a network (i.e. a graph or directed graph) from some given source node $s_i$ to a given destination node $t_i$. The resources are the edges in the network.

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For the particular example (4.4.1) in Chapter 4, the defined congestion cost on an edge $e$ to any driver $i$ using a path $P_i$ that contains $e$ is some function $c_e(n)$ of the number $n$ of drivers choosing a path containing $e$. (Although not necessary, it is usually assumed that $c_e()$ is non-decreasing.) The total congestion cost to a driver is the sum of edge congestions incurred by the driver. That is, for a strategy profile $\mathbf{P} = (P_1, \ldots, P_k)$, we have:

$$cost_i(\mathbf{P}) = \sum_{e \in P_i} c_e(n_e(\mathbf{P}))$$

where $n_e(\mathbf{P}) = |\{j : e \in P_j\}|$; i.e. the number of drivers using edge $e$.

# Potential games

We will now see that this congestion game (and more generally every congestion game) is a *potential game*.

**Potential games**

In a $k$-player potential function game with strategy profiles $\{S_1, \ldots, S_k\}$, there is a *potential function* $\psi : S_1 \times S_2 \cdots \times S_k \to \mathbb{R}$ satisfying:

$$\psi(s_i, \mathbf{s}_{-i}) - \psi(s_i', \mathbf{s}_{-i}) = u_i(s_i, s_{-i}) - u_i(s_i', \mathbf{s}_{-i})$$

for all $s_i, s_i', \mathbf{s}_{-i}$.

In our congestion game, the potential function is :

$$\psi(\mathbf{P}) = \sum_e \sum_{\ell=1}^{n_e(\mathbf{P})} c_e(\ell)$$

# Observations regarding the congestion game potential function

$$\psi(\mathbf{P}) = \sum_{e} \sum_{\ell=1}^{n_e(\mathbf{P})} c_e(\ell)$$

- This potential function on $k$ strategies naturally induces a potential function on $k-1$ strategies.
- This function does not depend on the order in which players arrive.
- Furthermore, it can be shown that:

$$\psi(\mathbf{P}) = \psi(\mathbf{P}_{-i}) + cost_i(\mathbf{P}) \text{ for all } i$$

.

Indeed this last property is common to all potential functions, namely:

$$\psi(\mathbf{s}) = \psi(\mathbf{s}_{-i}) + u_i(\mathbf{s}) \text{ for all } i$$

# Potential games: the good

The good news:

**Finite potential games have pure NE**

In any finite potential game, *repeated (better) play dynamics* must convege to a pure Nash equilibrium. That is, if a potential game is played in rounds or steps, and in each round, any player who can improve their utility by changing to a different pure strategy does so, then this process will converge to pure NE in a finite number of steps. In particular, best response dynamics will converge to a pure NE.

Proof sketch: The potential function is non-negative. For a cost problem, each improvement step lowers the cost to an agent and therefore decreases the potential function. This insures the process must converge tp a pure Nash.

For a maximization problem, the fact that there are only finitely many configurations, and that each improvement step is increasing will imply covergence to a pure Nash.

# The bad news and a few more comments about potential games

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For some special cases of potential games, there are polynomial time algorithms to find a pure Nash but in general there is again evidence (like for games in general) that the problem is hard (i.e. might require exponential time).

One final comment (for now) regarding potential functions is that there is a converse to the fact that every congestion game is a potential game. Namely, for every potential function game, there is a congestion game with the same potential funtion.

# An infinite road network game

We consider consider a network with a single source and destination and the goal is to send one unit of flow from $s$ to $t$. The infinite strategy space is to send fractions of flow along different paths.

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In this case, we can think of having some large number $k$ of drivers each *selfishly* choosing a path so as to minimize its time to arrive at the destination. For $k = \sum_i k_i$, if $k_i$ drivers choose some path $P_i$, then we can let $f_i = k_i/k$ be the fraction of drivers choosing path $P_i$. As $k \to \infty$, we are limiting to the infinite continuous game.

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We are interested in how much does the social welfare (i.e. the sum of all driving times) suffer from such selfish routing.

# The infinite network game with linear latency (i.e. cost) functions

We will consider linear latency functions where the delay of traversing an edge $e$ with a fraction $x$ of the unit flow is defined as an affine linear function $\ell_e(x) = a_e \cdot x + b_e$ where we can assume that the scalars $a_e$ and $b_e$ are non negative so that $\ell_e(x)$ is non-negative and non-decreasing. In particular, the delay on an edge could be a non-negative constant (when $a_e = 0$) not depending on the fraction $x$ or the delay will increase with $x$ if $a_e > 0$.

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Surprisingly, for such networks, adding "fast" additional roads (i.e. edges) can increase the social welfare (i.e. the sum of driving times) at equilibrium when drivers are selfish agents. This is known as the Braess Paradox. Equivalently removing a road can improve the social welfare at equilibrium.

# The Braess paradox and the price of anarchy

We have already introduced the *price of anarchy* as the worst case ratio of the social welfare cost at equilibrium to the cost of an optimal solution. The Braess paradox shows that this price of anarchy for linear latency function is at least $\frac{4}{3}$ and this holds for a very simple network example.

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We will also show that this simple network exhibits the worst possible price of anarchy.

As the text poins out, this seemingly peculiar game theory phenomena played out in New York City when the city decided to close one of its most congested streets. Instead of causing chaos, the total travel time (and hence the average travel time for drivers) actually improved!

# The impact of adding a super fast road to a simple network

Consider the following simple network in Figure 8.1 of the KP text with the indicated latency functions.

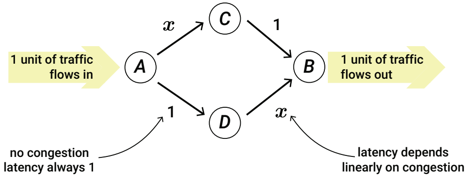


**Figure :** The initial network in the Braess paradox

In order to have an equilibrium, it should be clear that the total travel on these two disjoint paths must be equal. It follows that there is a unique equilibrium, namely when $x = \frac{1}{2}$ so that the social welfare is $\frac{3}{2}$.

# The impact of adding a super fast road to a simple network conitnued

Now as in Figure 8.2 of the KP text, consider adding a super fast road between nodes $C$ and $D$.
Note: Setting the road latency to 0 is just a convenience setting the latency to some relatively small $\epsilon$.

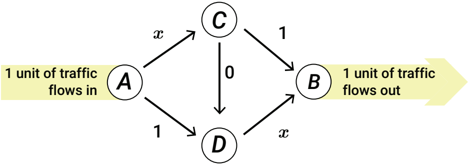


**Figure :** The network after a super fast road has been added

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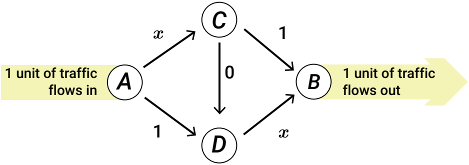


**Figure :** The network after a super fast road has been added

Now what equilibrium results? It is not difficult to see that the unique equilibrium is now for every driver to follow the path $A \rightarrow C \rightarrow D \rightarrow B$ (i.e. setting $x = 1$) with social welfare 2. POA $= \frac{4}{3}$..