CSC304: Algorithmic Game Theory and Mechanism Design
Fall 2016

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Lecture 2

Announcements

- While we have a choice of rooms with full teaching stations for Wednesdays and Fridays, unfortunately there are some negatives. Hence we will stay in the rooms assigned to us. Namely, SS 1087 on Mondays, and SS 1086 on Wednesdays and Fridays.
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- **Talk of possible interetst**: This coming Tuesday, September 20, there will be a seminar "Preferences and Manipulative Actions in Elections" by Gabor Erdelyi. The talk will be held in Pratt 266.

Agenda for today and next few lectures

We ended the first class asking "What kind of games are we considering?"

- We begin the technical discussion (this lecture and probably the next) with material from Chapter 4. More specifically we will first discuss two person (general-sum) games (in matrix or normal form). We do this while introducing some of the fundamental concepts in game theory that apply more generally. In particular, we will introduce pure and mixed Nash equilibrium.
- We will return to games with more than two players after discussing material from Chapter 2 (and one section in Chapter 3). I suggest first reading Chapter 4 up to and including Section 4.3. After our discussion of material in Chapters 2 and 3, we will finish the remaining sections in Chapter 4.
- Chapter 2 considers an important class of games, namely two person zero-sum games. For Chapter 2, read the chapter with the exception of Section 2.6. For Chapter 3, read the Section 3.2.

Many types of games

- Perhaps the most common perception of games are two person games such as chess, tic-tac-toe, etc and now more in vogue video games. These can be thought of as games played in alternating (or random) rounds and usually the utility to a player (i.e. an agent) is just some value associated with winning. Such games are examples of what are called games in *extensive form* and that is the subject of Chapter 6.
- We will start the course by looking briefly at *finite* games played *simultaneously* by two or more (finitely many) agents, each agent having some finite number of *strategies* and a real valued *utility* for the various possible outcomes of the game. This topic is called general-sum games in *normal form*. They are also called games in *matrix form* or *standard form*.
- Normal form games can be studied as "one-shot" games or as repeated games.
- The utilities represent agents/players who can be cooperative (to some extent) or competitive. In all cases, the goal of a player is to optimize their utility.

The Stag-Hunt game: An example of a two person game played simultaneously

Although games can involve many players, often examples are given for two player games because they are easier to illustrate. Furthermore, to simplify things even more we will see many examples where each player has only two strategies.

The Stag-Hunt narrative: example 4.1.2 in KP text)

Two hunters are stalking a stag and a hare and have to act immediately (and can't discuss what to do) or both the stag and hare will escape. The stag provides 8 days of food for a hunter and a hare only 2 days. But to catch the stag, both hunters are needed and they will then have to share the value of the stag. On the other hand each hunter can catch the hare on their own or share it if they both decide to go for the hare.

How to state this game precisely?

Matrix (also called normal or standard) form games

In order to precisely formulate such a two person game, we use a matrix. The rows of the matrix correspond to the strategies of the first agent (i.e. Hunter I) and the columns correspond to the strategies of Hunter II. (For two person games with more strategies per agent, we would use an $m \times n$ matrix.) The matrix entries specify the payoffs to the players; the first (resp. second) component being the payoff for Hunter I (resp/ Hunter II).

		Hunter II	
		stag (S)	hare (H)
Hunter I	stag (S)	(4, 4)	(0, 2)
	hare (H)	(2, 0)	(1, 1)

Figure: The stag-hunt payoff matrix; Figure 4.2 in KP text

Some concepts and observations about this game

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- The Hare 'H' strategy is a *safe strategy* for each player in that playing H will guarantee the best (i.e. maximum) minimum payoff when minimizing over all possible strategy profiles for the other player(s).
- In this game, neither player has a *dominant strategy*. A *(weakly) dominant strategy* D for a player P is one in which P will achieve the maximize payoff possible for any given strategy profile of the other players. (Usually one assumes that D is better for P for at least one strategy profile of the other players. If two strategies have the same value for all strategy profiles of the other players, then we do not have to distinguish these two strategies.) A *strictly dominant strategy* is one that yields a better value for all strategy profiles of the other players.

More observations about this game

- A dominant strategy is a safe strategy but not necessarily conversely. Note that H is not a dominant strategy.
- Given the strategy pair (H,S), the *best response* of player I is to change to strategy S. Similarly, given the strategy pair (S,H), the best response of of player II is also to change to strategy S.
- The strategy pair (S,S) is an *optimal strategy profile*; that is, this pair of strategies maximizes the *social welfare* defined as the sum of the player payoffs.

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- The strategy pairs (H,H) and (S,S) are *pure Nash equilibria (NE)*. That is, assuming the other player(s) do not change their strategies, each player cannot benefit (and here will lose) by changing its strategy. Equivalently, in a NE, a best response of each player is not to change strategies. (There can be more than one best response.)

How general are the previous observations about the stag-hunt game?

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- Given any strategy profile (i.e. pair of strategies in a 2-person game), each player always has at least one best response.
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- In some games, there are dominant strategies for some or all players. More generally, (i.e. for possibly more than two strategies per player), a given strategy s_i might dominate a strategy s_j in which case, we can eliminate strategy s_j and simplify the game.
- Not all games have pure Nash equilibria.

A game without any pure Nash equlibria

		Inspector	
		Don't Inspect	Inspect
Driver	Legal	(0, 0)	(0, -1)
	Illegal	(10, -10)	(-90, -6)

Figure: Driver and parking inspector game; Example 4.1.4 in KP

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However, the seminal result of Nash is that : Every finite general-sum game has a (possibly mixed) Nash equilibrium; that is, independently players probabilistically choose strategies so that their *expected payoff* is maximized assuming all other players are playing their (possibly mixed) equilibrium mixed strategy.

Some games have both pure and mixed Nash equilibria.

The mixed strategy for the stag-hunt game

For the stag-hunt game each player will play strategy S with probablility $\frac{1}{3}$ and strategy H with probability $\frac{2}{3}$ resulting in each player obtaining expected payoff $\frac{1}{9} \cdot 4 + \frac{2}{9} \cdot 2 + \frac{2}{9} \cdot 0 + \frac{4}{9} \cdot 1 = 4/3$.

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How do we verify or find such a mixed NE?

It is easy (i.e. polynomial time) to verify that a strategy profile is a pure or mixed NE; namely, just see if a best response for each player is to not change their strategy. More specifically, whenever the other player(s) have set their (pure or mixed) strategies, a best response is always a pure strategy. Hence it suffices to compute the (expected) payoff for each player P and verify that it is at least as good as each pure strategy for P.

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However, **we now "have evidence"** from a sequence of results in complexity theory (see references in the notes of Chapter 4) that even for two player games (with each player having many strategies) that (in the worst case) **it is computationally difficult (e.g. possibly exponential time) to find a mixed NE**.

Efficiently finding a mixed NE for a “2 by 2” two person game

For a “2 by 2” two person game (such as the stag-hunt game) where each player has only two strategies, using the *principle of indifference* it is relatively easy to understand how to verify or find a mixed NE as follows: Suppose, in a mixed NE, player 1 (resp. player 2) uses the mixed strategy defined by the probability vector $\mathbf{x} = (x_1, x_2)$ (resp. defined by the probability vector $\mathbf{y} = (y_1, y_2)$.

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Given that we have a NE, when player I is playing mixed strategy $\mathbf{x}$, it must be that the expected payoff for player II must be the same for each of its pure strategies or else the best response for player II would be to purely play the better strategy against player 1’s mixed strategy. Similarly, for player I, playing against player II’s mixed strategy.

The mixed strategy for the stag-hare game

Since this is a symmetric game, we need only consider one player, say player 1. So again assume player 1 is playing the mixed strategy $\mathbf{x}$.

For the stag-hunt game, we have

1. The expected payoff for player II is $x_1 \cdot 4 + x_2 \cdot 0)$ if player II plays strategy S.
2. The expected payoff for player II is $x_1 \cdot 2 + x_2 \cdot 1)$ if player II plays strategy H.
3. Setting these two expectations to be equal, we have :

$$4x_1 = 2x_1 + x_2$$

which along with $x_1 + x_2 = 1$ implies that $x_1 = \frac{1}{3}$ and $x_2 = \frac{2}{3}$.

The principle of indifference

For two person games, we can state the principle of indifference as follows:

Principle of indifference

In any m by n game two person game, suppose there is a mixed NE where the row player (resp. column) player is using mixture $\mathbf{x}$ (resp. $\mathbf{y}$). If $I = \{x_i > 0\}$ and $J = \{y_j > 0\}$ then for all $i \in I$, $\mathbb{E}_{\mathbf{x}}$ (i.e. the column players expected payoffs wrt $\mathbf{x}$) for the pure strategies in J must be equal. Similarly the expected payoffs (wrt $\mathbf{y}$) for the pure strategies in I must be equal.

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Consider how you can use the principle of indifference to find all possible Nash equilibiria for a two person game when each player only has a "small" number of strategies.

Note: There can be a continuum (i.e. a subspace) of mixed NE. Note: The principle of indifference provides necessary conditions for a mixed NE but one still have to verify that each candidate is an NE.