CSC304: Algorithmic Game Theory and Mechanism Design
Fall 2016

Allan Borodin (instructor)
Tyrone Strangway and Young Wu (TAs)

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Lecture 14

- Announcements
  - Office hours: Tuesdays 3:30-4:30 SF 2303B
    or schedule meeting; or drop by.
  - Term test will take place on Friday, November 4 (not Nov. 5). There was a typo in the course information sheet.
    Aids: One sheet of handwritten notes, both sides.
  - Scope of term test:
    - Any topic relating to questions that were asked in the first two assignments. In particular, this includes: basic game theory, games in normal form and extensive form, pure and mixed NE, network congestion and Braess paradox, Bayesian mechanism design for maximizing revenue in single item auctions; Myerson auction.
    - Material covered in lectures 12,13 and todays lecture 14 on mechanisms VCG and GSP for sponsored search.
  - No class on Monday, November 7 (fall break)
- Todays agenda
  - Review of VCG for sponsored search; VCG is envy-free.
  - GSP for sponsored search
  - The public project problem

Returning to VCG and the sponsored search problem

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If bidders are truthful, then both VCG and GSP maximize social welfare. We know that VCG is a truthful mechanism so lets first consider VCG for the sponsored search problem.

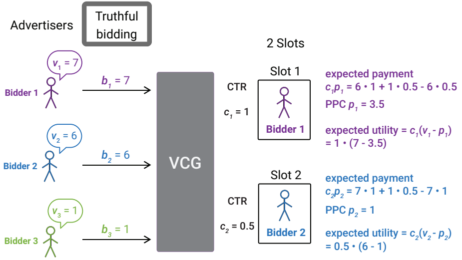

Figure: Figure 15.7: The depiction of VCG for an example of sponsored search

VCG for sponsored search

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Consider the price for the advertiser winning the ith slot for $i \leq k$. The impact on the social welfare by the ith advertiser is to push advertisers $j > i$ down one slot. That is, he imposes an (expected) cost to the jth advertiser of $b_j(c_{j-1} - c_j)$ for a total impact of $\sum_{j=i+1}^{k+1} b_j(c_{j-1} - c_j)$.

Hence we want to charge the ith advertiser a per click bid $p_i(\mathbf{b})$ so that his expected cost is equal to the impact on the other advertisers; that is,

$$c_i p_i(\mathbf{b}) = \sum_{j=i+1}^{k+1} b_j(c_{j-1} - c_j) \quad \text{so that}$$

$$p_i(\mathbf{b}) = \sum_{j=i+1}^{k+1} b_j \frac{(c_{j-1} - c_j)}{c_i}$$

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In the example, the values (= bids) are $b_1 = 7, b_2 = 6$ and $b_3 = 1$ and there are 2 slots with CTR $c_1 = 1$ and $c_2 = .5$. Bidder i obtains slots i for $i = 1, 2$.

1. With bidder 1 being present, the expected social welfare of the remaining two bidders is $c_2 \cdot 6 + 0 = 3$.

   If bidder 1 were not present, the expected social welfare for the two remaining bidders would be $c_1 \cdot 6 + c_2 \cdot 1 = 6.5$. So bidder 1 has caused them a loss of social welfare of $6.5 - 3 = 3.5$.

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   The PPC p_1 satisfies $c_1 \cdot p_1 = 3.5$ so that $p_1 = 3.5$.
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   The PPC p_2 for bidder 2 satisfies $c_2 \cdot p_2 = 0.5$ so that $p_2 = 1$.

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When all slots have the same click through rate, this is just a multi-unit auction for a given item (i.e. selling k identical items).

VCG is envy-free for sponsored search

Intuitively, a mechanism is *envy-free* if it always results in an outcome in which each agent does not envy what some other agent has obtained.

In the KP text, this concept is first discussed in Chapter 11 (fair division, cake cutting) which concerns *mechanisms without money*.

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With regard to auctions, a mechanism is envy-free if for all i and for all allocations 'a' and payments 'p', agent i obtaining a_i does not prefer (in terms of utility) the allocation and payment obtained by some other agent $j \neq i$; that is, $v_i(a_i(\mathbf{b})) - p_i(\mathbf{b}) \geq v_i(a_j(\mathbf{b})) - p_j(\mathbf{b})$

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VCG is envy-free for the k slot sponsored search problem. Namely, $c_i(v_i - p_i) \geq c_j(v_i - p_j)$ for all $i \neq j$. (Here, we assume $c_j = 0$ for $j > k$.)

The proof is essentially in seeing how to view VCG as VCG applied to k different single item auctions. VCG (i.e. Vickrey) is easily seen to be envy-free for a single item.

GSP for sponsored search

Like VCG, the GSP mechanism awards the ith slot to the ith highest bidder for $1 \leq i \leq k =$ number of slots; the PPC for the ith winner is the next PPC bid b_{k+1}.

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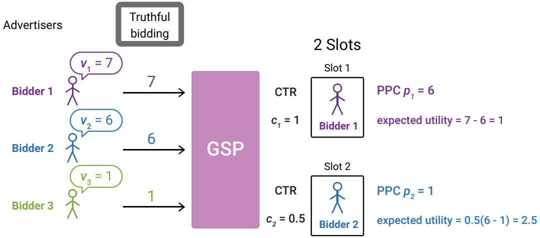
Consider GSP applied to this example. If bidding truthfully, the highest valued agent will have expected utility $1 \cdot (7 - 6) = 1$ and the second highest agent will have utiltiy $.5 \cdot (6 - 1) = 2.5$ as in VCG. .

But even with the smaller CTR, the highest valued agent would do better to reduce their bid to say 5 so as to obtain slot 2 since that would yield utility $.5 \cdot (7 - 1) = 3$ which is much better than truthful bidding.

Figure 15-9; GSP is not truthful



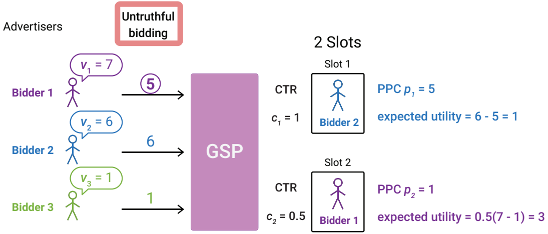


Figure: Figure 15.9: Bidding truthfully need not be an equilbrium for GSP

VCG vs GSP

So what mechanism should a search engine use for sponsored search?

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But there are many other considerations, not the least of which is that a search engine may be more interested in revenue than social welfare. (There is an argument to be made for social welfare since this game will be played repeatedly and agents may like to know that social welfare is being maximized and hence may be more willing to advertise more.)

VCG vs GSP: What makes more revenue?

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As before we assume values $v_1 \geq v_2 \ldots v_n$ and CTRs $c_1 \geq c_2 \ldots c_k$ for the k slots. Hence a social optimal allocation is for agent i to receive slot i.

Lemma 15.5.2 shows that for every instance of sponsored search, there is always a GSP equilibrium which obtains the same (socially optimal) allocation and the same revenue as in the VCG mechanism.

Achieving the truthful VCG revenue and allocation in a GSP equilibrium

Let p_i^{VCG} for agent i denote the VCG price for an instance of sponsored search. Then bidding $b_1 > p^{VCG}$, and $b_i = p_{i-1}^{VCG}$ for $2 \leq i \leq k$ is an equilibrium that (clearly) has the same allocation and revenue as VCG.

The only thing that has to be shown is that this bidding is an equilibrium. This follows from the fact that the VCG allocation is envy free.

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1. With bidder 1 being present, the expected social welfare of the remaining two bidders is $c_2 \cdot 6 + 0 = 24$.

   If bidder 1 were not present, then the the expected social welfare for the two remaining bidders would be $c_1 \cdot 6 + c_2 \cdot 1 = 64$. So bidder 1 has caused them a loss of social welfare of $64 - 24 = 40$ and that is the expected revenue generated from bidder 1.
2. With bidder 2 present, the expected social welfare for bidders 1,3 is $c_1 \cdot 7 + 0 = 70$

   If bidder 2 were not present then the social welfare for the two remaining bidders would be $c_1 \cdot 7 + c_2 \cdot 1 = 74$. So bidder 2 has caused them an expected loss of social welfare of $74 - 70 = 4$.

Hence the total expected revenue generated by VCG is $40 + 4 = 44$

GSP with more and less that the expected VCG revenue

We have just seen that VCG generates expected revenue 44.
We now provide two other NE for this instance, one with more revenue and one with less.

- Bidding $(b_1, b_2, b_3) = (5, 4, 2)$ is a GSP equilibrium. It clearly achieves the same social optimal allocation but now with greater revenue $4 \cdot 10 + 2 \cdot 4 = 48$ than VCG.
- Bidding $(b_1, b_2, b_3) = (3, 5, 2)$ is a GSP equilibrium. It clearly does not achieve the same social optimal allocation and obtains less revenue $3 \cdot 10 + 2 \cdot 4 = 38$ than VCG.

Sponsored search as a combinatorial auction

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What is GSP pricing? GSP is a critical price mechanism. It is in some sense a single parameter problem (i.e. each agent has only a single private value) but not in the sense of Chapter 15 (where the value for a winning agent does not depend on the allocation).

Moreover, as a CA, it is *not* a single-minded CA. Each agent is interested in k different slots so we cannot appeal to the truthful Mu'alem and Nisan monotone greedy allocation mechanism using critical prices.

Some differences between this simplifed view of search engine advertising and the "real problem"

- Given the importance of sponsored search, there are various studies as to whether or not VCG or GSP is better "in practice". I do not think there is a clear consensus. Of course, one would need access to extensive real data revenue obtained in some controlled experiment.
- Advertisers also have (say daily) budgets.
- How does click through rates translate into sales? Could it be that users clicking on the second slot are more likely buyers?
- Search engines do not just take the bids as input. Rather the search engine computes a quality factor q_i that measures the match between the advertisement of agent i and the search request. Then v_i is replaced by $v_i' = v_i q_i$.
- We are not taking into consideration how competing or complimentary advertisements will impact the ultimate benefit of obtaining a slot. That is, as in all CAs, we assumed no externalities.

The public project problem and VCG

In section 15.4.3, the public project problem is presented as a decision problem. Here the government wants to build some public facility (e.g. a swimming pool, a library, a bride) and has a public budget C that it plans to spend on this project if the project is sufficiently valuable to enough people. (More generally, it might have a subset of projects and wants to choose a subset of projects that can be done within the budget.)

Suppose individuals have a private value of v_i for the project. The government will undertake the project only if the $\sum_i v_i \geq C$. This is then a single parameter problem where the feasible allocations are those that achieve $\sum_i v_i \geq C$.

Note: This is not a combinatorial auction. Either everyone is a winner when the project is done or no one is a winner in which case the social welfare is 0.

Public project problem continued

In order to solicit honest valuations, the government could run a VCG auction. That is, ask everyone for their value, do the project if the social welfare is at least C, and then price each person's contribution (if it is needed) as the marginal value they provide to the project. That is, the price is $p_i = C - \sum_{j \neq i} v_j$ if $\sum_{j \neq i} v_j < C$.

This will be truthful but it could lead to no payments and to collusion.

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This general optimization problem is called the *combinatorial public projects problem (CPPP)*.

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For $k = 1$, the KP text has an example showing that (in an informal sense), the VCG auction would not not be envy-free.