Assignment 3: sample solutions

CSC 304F: Algorithmic game theory and mechanism design

Due: Noon, Wednesday, November 30, 2 016. Note that this assignment was initially due on November 23 and then extended to November 30 to give more time for topics being discussed. Due to Markus being unavailable for a period of time, we further extended the due date until Friday, December 2.

1. Consider the following combinatorial auction with 2 agents and 2 objects $\{a, b\}$. The valuation of agent 1 is $v_1(\{a\} = 14, v_1(\{b\}) = 1$ and the valuation of agent 2 is $v_2(\{a\} = 10, v_2(\{b\}) = 10, v_2(\{a, b\}) = 30$.

   - What is the allocation and payments for the VCG mechanism.
     Solution: Agent 2 is allocated both objects and pays $14 - 0 = 14$. Agent 1 pays nothing.
   - Consider the greedy algorithm that sorts all the 5 individual bids $\{b_i(S_i)\}$ by non-increasing value of $r(i, S) = \frac{b_i}{|S|}$. For this CA, if truthful agent 2's bid for $S = \{a, b\}$ would come first and then agent 1's bid for $S = \{a\}$. The algorithm considers each bid in order and accepts the bid (i.e. allocates the requested set) if that set does not intersect with a previously allocated set and the agent has not already been allocated a set. Suppose that we have a mechanism that uses the above greedy algorithm and for each accepted bid (b_i, S_i) charges $|S_i| \cdot \frac{b_j}{|S_j|}$ where agent j is the next agent to be allocated a non-empty set S_j. (If S_i is the last set allocated, then agents i pays nothing. That is, each agent is being charged per item at the next lowest price per item.

     What is the allocation and prices for this greedy mechanism.
     Solution: There was an ambiguity is how we sgtated the pricing for this auction. I meant that that j is the next person to be allocated if you were not in the auction. This is the same as saying you pay per item the least that you could pay and still come before agent j. In this case, agent 2 is allocated both objects and now pays $2 \cdot \frac{14}{1} = 28$ thereby obtaining utility $30 - 28 = 2$. Agent 1 pays nothing. If you interpreted this as saying that $j$is the next person to be allocated even when you are there, then you pay nothing.
   - Is this mechanism truthful? If yes, indicate how truthfulness follows from some known result. If not, which agent could bid non truthfully so as to result in higher utility (say assuming the other agent bids truthfully).

     Solution: The ambiguity above leads to two different answers. The mechansim is not truthful for either interpretation. For the interpretation I had in mind, agent 2 can bid 26 for $\{a, b\}$ and 10 for $\{b\}$. Agent one will be allocated $\{a\}$ and agent 2 will be allocated $\{b\}$. Agent 2 pays nothing and therefore has utility $10 - 0 = 10$ and hence will underbid. For the other interpretation, agent 1 can bid 16 for item a and win the auction and receive utility $14 - 10 = 4$ which is better than not being allocated and haiving 0 utility.

2. In the lecture on Wednesday, November 9, I stated that using a potential argument, the ascending algorithm must terminate and will result in an allocation that is optimal for social welfare. Prove these statements. Hint: This is easy but not as easy as I first thought. I will explain in class (Monday, November 21) the one somewhat subtle point.

   Solution: We argue that the potential is decreasing in each round. This is because the price is raised by one unti for some k items while the utlility decreases for some $k+1$ buyers. The potential starts out as the sum of buyer values (i.e. a positive number). If the potential never becomes negative that would prove termination and hence prove that the everyone obtained an item in their demand set. However as the algorithm is stated, there is no reason that the potential is non-negative. But as we argued in class, we can reset prices each round by deducting the minimum of all the prices so that the potential will always be non-negative. Reducing every price by the same amount does not change the demand sets. So the algorithm has the same behavior with or without the price reduction.

3. Consider a matching market in which there are four condos, C1, C2, C3, and C4, for sale in a certain neighborhood. There are four buyers Dan, Etta, Fahiem, and Gabor wishing to purchase these condos. The buyers have the following valuations:

	C1	C2	C3	C4
Dan	2	7	6	4
Etta	7	9	5	4
Fahiem	3	2	5	4
Gabor	4	7	2	1

   Solution: This question was discussed in class, Wednesday December 7. Below, I will sketch the answer for the first part of the question.

   (a) Suppose we run the "auction" mechanism described in lectures 16 and 17 to determine market clearing prices, but *without using the price reduction step (i.e., we do not insist that the smallest price is* 0*).* as discussed in lecture 19. Recall that at each round of the auction, if there is a constricted set of buyers in the preferred seller graph, then we trigger price increases for one or more condos. We want you to run this auction using the following rule: *if there is more than one possible constricted set of buyers at any round of the auction, choose any* ***minimal*** *set of constricted buyers to determine which condo prices increase.* For each round of the auction, show each of the following: i. prices for each condo; ii. the preferred seller graph; iii. if the prices are not market clearing at that round, the *minimal* constricted set of buyers you selected and the condos whose prices will increase; and iv. if the prices are market clearing at that round, a perfect matching that assigns a condo to each buyer.

   Solution sketch:
   A minimal constricted set in round 1 is $\{E, G\}$ raising price of C2 by 1 A minimal constricted set in round 2 is $\{E, G\}$ raising price of C2 by 1 A minimal constricted set in round 3 is $\{D, F\}$ raising price of C3 by 1 The prices $C1 = 0, C_2 = 2, C3 = 1, C4 = 0$ result in an evefyone gettign a condo in their demand set.

(b) Repeat part (a), but this time, if there is more than one possible constricted set of buyers at any round of the auction, choose a **maximal** set of constricted buyers to determine which condo prices increase. Be sure to show all details of every round as in part (a). How do the market clearing prices reached compare to those in part (a)?

(c) Starting from the market clearing prices computed in part (a), suppose the sellers each decided to increase their prices by the same amount i. What is the largest price increase i they could impose without changing the preferred seller graph?

(d) Suppose another very attractive condo C5 comes onto the market. All four buyers value this condo more than any of the other four condos. The owner of this condo recognizing its attractiveness prices it higher than any of the buyer valuations. (i) Briefly explain why the addition of this condo (at this price) won't change the market clearing prices of the other condos. (ii) In real-world real estate markets, the entrance of a new condo into the market at a high price will often cause prices for other condos in the neighborhood to increase in value. Explain what feature(s) of the market clearing model we are consideruing prevents this from happening, and what feature(s) of real-world real estate might cause this to occur.

4. Consider a stable marriage problem with four women Xina, Yael, Zoe and Winnie, and four men, Arnold, Bikash, Cam, and Denzel. Using initials instead of names, the women's preference for the men are:

$$X : A \succ B \succ C \succ D \qquad Y : A \succ B \succ C \succ D \qquad Z : B \succ D \succ A \succ C \qquad W : C \succ A \succ B \succ D$$

And the men's preferences for the women are:

$$A : Z \succ W \succ X \succ Y \qquad B : X \succ Y \succ Z \succ W \qquad C : X \succ Y \succ W \succ Z \qquad D : Y \succ W \succ X \succ Z$$

Solution: This question was discussed in the Wednesday, September 7 class.

(a) Suppose we run the female-proposing deferred acceptance (FPDA) algorithm. Show how each iteration will proceed by: (i) clearly labelling each iteration; (ii) stating exactly which proposals will be made at that iteration; (iii) and stating exactly which engagements will be in place at the end of that iteration (once relevant proposals are accepted or rejected). Indicate clearly which iteration is the final one and what stable marriages result from FPDA.

Solution sketch:
Y gets jilted by A in first round.
Z gets jilted by B in second round.
Final set of proposals is a stable matching $X : A, Y : B, Z : D, W : C$

(b) We say a man *lies* in FPDA when he rejects a proposal from a woman even though he prefers the proposer to his current fiancee. Identify some man who can *lie* by falsely rejecting a proposal, thereby changing the outcome of FPDA so that he ends up married to a more preferred partner that he did in part (a). State when he should lie in FPDA and what stable marriages will result. (Assume all other men continue to accept and reject proposals truthfully.)

Solution sktech: It is easiest to first do the MPDA to see who is better off in the MPDA, For example, in the MPDA, A is matched with Z who he prefers to his match in the FPDA. If A

changes his preference ranking to $Z \succ W \succ Y \succ X$ he will wind up with his prefered partner Z.

(c) Now consider the *male-proposing* deferred acceptance (MPDA) algorithm on the same problem (i.e., same set of preferences). As in part (a), clearly indicate what happens at each iteration of MPDA and what stable marriages result when MPDA terminates.

5. Suppose we run a plurality election to determine a winner from a set of four candidates a, b, c, d. Assume that any ties are broken in alphabetical order (e.g., if a and c tie with the most votes, then a is declared the winner). Consider the following profile of eight voter preferences. For convenience, the plurality scores for each candidate are also shown (but you're encouraged to verify them):

Voter	V1	V2	V3	V4	V5	V6	V7	V8
more preferred	c	c	b	b	a	a	c	c
↓	b	b	a	c	b	d	b	b
	d	a	d	d	c	b	a	d
less preferred	a	d	c	a	d	c	d	a

	Plurality Score	Borda Score
a	2	10
b	2	17
c	4	15
d	0	6

bf Solution: This question was also discussed in the Wednesday, December 7 class.

- Assuming everyone votes sincerely. Can any single voter change her vote (assuming everyone else votes sincerely) and change the result of the plurality election so that she prefers the new winner to the original winner c? If so, briefly explain how (who should change her vote, how she would change it, and who the new winner would be). If not, briefly explain why not.
- Now suppose we use the Borda rule for this election. The Borda scores, assuming sincere voting, are shown above. (Again, you are encouraged to verify the scores.). The winner would now be b rather than c. The four voters $(V1, V2, V7, V8)$ whose favorite candidate is c do not know how the others will vote, but strongly suspect that b is very popular. With this suspicion, how do you think they should report, or *misreport*, their preferences to increase the odds of their favorite candidate c winning? If they all vote this way, will c win? (Assume the other four voters vote sincerely.)

  Solution sketch: Voters $V1, V2, V7, V8$ should clearly move b to be last in thier preference rankings.
- The government suddenly discovers that there was an additional voter V9 whose ranking (and vote) was not counted. The preference ranking for V9 is $b \succ c \succ a \succ d$. In fairness, the government decides to include V9s preferences and now decides to use the single transferable vote (STV) rule for voting (using plurality to drop off the candidate with the lowest score and breaking ties in favour of the candidate who wins the most pairwise comparisons). Assuming all nine voters will vote sincerely, determine the winner of this election. Describe each round in this election.

  Solution sketch:
  After round 1, D is eliminated.
  After round 2, a is eliminated.
  The winner b is determined by the tie-breaking rule.