Assignment 1: Sample Solutions

CSC 304F: Algorithmic game theory and mechanism design

Due: October 7, 2016

Be sure to include your name, student number and tutorial room with your assignment. If your handwriting is *possibly* illegible, be sure to hand in your assignment in some typed form.

1. Consider the following game in matrix form with two players. Payoffs for the row player Shelia are indicated first in each cell, and payoffs for the column player Thomas are second.

	C	D
A	$10, 16$	$14, 24$
B	$15, 20$	$6, 12$

(a) Does either player have a dominant strategy? Explain your answer.

Answer: Neither player has a dominant strategy. For example, if Shelia plays A and Thomas plays D then Shelia's payoff is 14. But if Shelia plays B and Thomas plays C, then Sheilas's payoff is 15. A similar argument shows that Thomas also does not have a dominant strategy.

(b) What are the *pure* strategy Nash equilibria of this game? Justify your answer. If there is more than one pure equilibria, which would Thomas prefer? What is the Price of Anarchy (with respect to pure NE) for this game?

Answer: The two pure NE are:

- (B, C): If Shelia plays B, then Thomas' best response is C with a payoff of 20 rather than 12 if Thomas played D; if Thomas plays C, then Shelias's best response is B with a payoff of 15 rather than 10 if she plays A.
- (A, D): Similar reasoning shows that (A, D) is a pure NE.

The worst NE is (B, C) with SW of 35 while the optimum outcome is (A, D) with SW of 38. Thus, the pure PoA is 38/35.

(c) This game has a *fully mixed* strategy Nash equilibrium in which both Shelia and Thomas play each of their actions with positive probability. What are the mixed strategies for each player in this equilibrium? Show how you would compute such a mixed equilibrium and verify that your mixed strategies are indeed in equilibrium.

Answer: Let Thomas play C with probability p and hence play D with probability $(1 - p)$. By the principle of indifference, it must be that the expected payoff for Shelia is the same whether she plays A or B. Thus:

$$p(10) + (1 - p)(14) = p(15) + (1 - p)(6)$$

so that $p = \frac{8}{13}$.
Let Shelia play A with probability q and hence plays B with probability $(1-q)$. In order to insure that Thomas' payoff's are the same for strategies C and D, we have that:

$$q(16) + (1-q)(20) = q(24) + (1-q)(12)$$

so that $q = \frac{1}{2}$.
To verify that $p = \frac{8}{13}, q = \frac{1}{2}$ is a mixed NE, we calculate

- Shelia's payoff for A is $\frac{8}{13}(10) + \frac{5}{13}(14) = \frac{150}{13}$
- Shelia's payoff for B is $\frac{8}{13}(15) + \frac{5}{13}(6) = \frac{150}{13}$
- Thomas' payoff for C is $\frac{1}{2}(16) + \frac{1}{2}(20) = 18$
- Thomas' payoff for D is $\frac{1}{2}(24) + \frac{1}{2}(12) = 18$

(d) Suppose that Shelia and Thomas play this game repeatedly once per day, each time choosing their actions according to some strategy. Shelia claims that she will play her mixed strategy according to the probabilities you calculated in part (c). Thomas decides to take Shelia at her word and after a few days commits to play a pure strategy from now on. Does it matter which one he plays and if so which one will he play?
Answer: By the principle of indifference, it does not matter which strategy Thomas plays.

(e) After Thomas commits to play a pure strategy as in part (d), should Shelia reneg on her word and play a different strategy knowing that Thomas has committed to a pure strategy and will never change? Why or why not?

Answer: For example, suppose Thomas now commits to pure strategy D and now Shelia notices that Thomas is always playing strategy D. Shelia will obviously be better off playing strategy A and hence they will arrive at the pure NE (A, D). Why would Thomas announce his commitment to D?
In this pure NE, Thomas' payoff is 24 whereas in the mixed NE, his strategy was only 18.

2. (a) Consider an arbitrary 2 by 2 two person game in which neither player has a dominant strategy. Prove the following: In such a game, every NE is either a pure NE or a fully mixed NE.

Answer: Consider an arbitrary 2 by 2 two person game in which no player has a dominant and without loss of generality assume that there is a *not* fully mixed NE where the column player purely plays C (i.e. with probability 1) and the row player and the row player plays A with probability q such that $0 < q < 1$. Then by the principle of indifference it must be that $a_1 = a_2$. Now assume without loss of generality that $c_2 \geq c_1$. Then strategy B is a dominant strategy for the row player contradicting our assumption.

	C	D
A	a_1, b_1	c_1, d_1
B	a_2, b_2	c_2, d_2

(b) Show that the above statement is not necessarily true for every 2 by 3 two person game; that is, exhibit a 2 by 3 game with no dominating strategies that has an NE with one player playing a pure strategy and the other playing a (non pure) mixed strategy.

Answer: Here is an example of where the statement fails:

	C	D	E
A	$2, 2$	$1, 2.5$	$2, 0$
B	$2, 1$	$2, 0$	$1, 2.5$

You can verify that ia nash equilibrium is obtained when the column player uses the pure stratgey C, and the row player chooses strategies A and Ba each with probability $\frac{1}{2}$.

3. Consider the following modification/clarification of exercise 2.12 in the KP text. Company I chooses one of the three locations (L,C,R) and simultaneously Company II places two restaurants at two different locations. While Company II will not place two restaurants in the same location, it is possible that both companies will have a restaurant in the same location.

This is being phrased as a zero-sum game where the value of the game is the probability that a customer visits the restaurant of Company I. Without loss of generality we can assume that a customer always knows which location is closest (i.e. does not have to worry about two locations being at the same distance).

Answer: Note that Company II chooses two different locations, at most one location may not have a restaurant. We observe that a customer will choose the Left (L) and Right (R) locations *if there is a restaurant there*, each with probability $\frac{1.5}{4} = \frac{3}{8}$ and location C with probability $\frac{1}{4}$. If there is no restaurant at L then, customers will choose location C with probability $\frac{5}{8}$ and location R with probability $\frac{3}{8}$. Similarly (by symmetry) if there is no restaurant at R, then customers will choose location C with probability $\frac{5}{8}$ and location L with probability $\frac{3}{8}$. Finally, in terms of which location customers will choose, if there is no restaurant at location C, they will choose L (and R) with probability $\frac{1}{2}$. The three strategies available to Company II are the location pairs (L, C). (L,R) and (C,R).

(a) Express this as a 3 by 3 zero-sum game in matrix form.

Answer: We claim the following three by three matrix expresses the zero-sum game.

	(L, C)	(L, R)	(C, R)
L	$3/16$	$4/16$	$3/8$
C	$5/16$	$4/16$	$5/16$
R	$3/8$	$4/16$	$3/16$

Lets first consider the probabilities for player I playing L, for each of the possible strategies of player II.

- If player II selects (L,C), then there is no restaurant at location R and hence customers will choose location L with probability 3/8 and so that the payoff (i.e. the probability) is 3/8.
- If player II selects (L,R), then there is no restaurant at location C and hence customers will choose location L with probability 1/2 and will choose each of the two restaurants at L with probability 1/2 so that the payoff (i.e. the probability) is 1/4.
- If player II selects (C,R), then there is a restaurant at all locations and and hence customers will choose location L with probability 3/8 and hence will choose Company I with that probability.

The situation is symmetric if player I chooses location R.

Finally we consider the probabilities when Company I chooses location C,

- If player II selects (L,C), then there is no restaurant at location R and hence customers will choose location C with probability 5/8 and will choose each of the two restaurants at C with probability 1/2.
- If player II selects (L,R), then there is a restaurant at all locations and and hence customers will choose location C with probability 1/4 and hence will choose Company I with that probability.
- If player II selects (C,R), then there is no restaurant at location L and hence customers will choose location C with probability 5/8 and will choose each of the two restaurants at C with probability 1/2.

(b) Express the value of the game as a LP maximization problem.

Answer: Maximize v Subject to
$$\frac{3}{16}x_1 + \frac{5}{16}x_2 + \frac{3}{8}x_3 \geq v$$
$$\frac{4}{16}x_1 + \frac{4}{16}x_2 + \frac{4}{16}x_3 \geq v$$
$$\frac{3}{8}x_1 + \frac{5}{16}x_2 + \frac{3}{16}x_3 \geq v$$
$$x_1 + x_2 + x_3 = 1$$

(c) Use the LP solver (link will be given) to solve the LP.

Answer: Using the LP solver, we change all coefficients in the constraints to be integers by multiplying by 16. (Note: There is probably a way to use the solver wth fractional coefficients.) Also change x_1, x_2, x_3 to x, y, z. The new equivalent LP is

Maximize v Subject to
$$3x + 5y + 6z \geq 16v$$
$$4x + 4y + 4z \geq 16v$$
$$6x + 5y + 3z \geq 16v$$
$$x + y + z = 1$$

The solution yields a strategy with x = 2/3, z = 1/3 and the value of the game v = 1/4. By symmetry x = 1/3, z = 2/3 is also a solution. Moreover, since there are only three equalities/inequalities in 4 unknowns, we can expect a subspace of solutions for x, y, z all yielding the same value v for the game. Indeed, any x, z with $x + z = 1$ (i.e. $y = 0$) is also a solution.

4. The prices and discounts for the Yoga classes are summarized as follows:

Number of Students	1	2	3
A's Per-Student Price	10	8	8
B's Per-Student Price	15	12	9

Three friends Xavier, Yasher, and Zoe are each considering taking one of the classes. But they each have different preferences or *valuations* for the classes. If they take a class that has valuation v and charges them a price p, then their utility or net *payoff* for that class is $v - p$. Their valuations are as follows:

	Xavier	Yasher	Zoe
Ashtanga	17	16	14
Bikram	20	18	21

(a) (i)

Z:A

	$Y:A$	$Y:B$
$X:A$	$9,8,6$	$9,3,6$
$X:B$	$5,8,6$	$8,6,4$

Z:B

	$Y:A$	$Y:B$
$X:A$	$9,8,6$	$7,6,9$
$X:B$	$8,6,9$	$11,9,12$

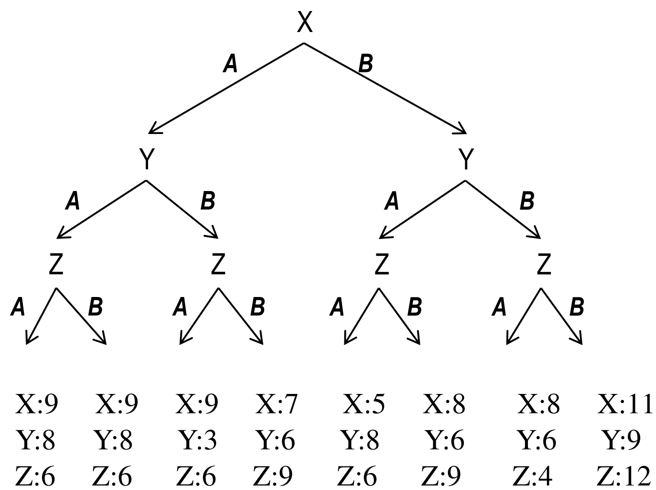
(ii) Which players (if any) have a dominant strategy in this game? Explain.
Answer: B is a dominant strategy for Z.
(iii) What are the pure Nash equilibria of this game? Are any of the pure Nash equilibria Pareto optimal?

Answer: The pure NE are the following: $(A, A, A), (A, A, B), (B, B, B)$. (B, B, B) is Pareto optimal.
(iv) What strategy profile has the highest social welfare (and is it a Nash equilibrium)?

Answer (B, B, B) is an optimal profile and it is a NE.

(b) (i) Draw the extensive form game tree for this game, assuming that players announce their choices in the following order: first Xavier chooses a class, then Yasher chooses, then Zoe chooses.

Answer: The tree for the extensive form game is as follows



(ii) Provide a description of the *subgame perfect equilibrium* in this extensive form game.

Answer: The only SGPE is (B, B, B)

(iii) Would the outcome of the game change if the order of the players' choices were reversed? Why or why not?

Answer: The SGPE remains (B, B, B)

(iv) Suppose that we changed Zoe's valuations as follows: Ashtanga now is worth 15 and Bikram is worth 15. Rewrite the extensive form game with these new payoffs (note: only Zoe's payoffs will change at each leaf of the tree). Describe an equilibrium in this new game that is *not* a subgame perfect equilibrium. Justify your answer.

Answer: The interesting thing is that with this change of valuations for Zoe, if she threatens to always select Ashtanga and that threat is believed then (A, A, A) becomes a NE giving Zoe more profit than the SGPE (B, B, B).