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The Design of Competitive Online Algorithms via a Primal–Dual Approach

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Abstract

The primal–dual method is a powerful algorithmic technique that has proved to be extremely useful for a wide variety of problems in the area of approximation algorithms for NP-hard problems. The method has its origins in the realm of exact algorithms, e.g., for matching and network flow. In the area of approximation algorithms, the primal–dual method has emerged as an important unifying design methodology, starting from the seminal work of Goemans and Williamson [60].

We show in this survey how to extend the primal–dual method to the setting of online algorithms, and show its applicability to a wide variety of fundamental problems. Among the online problems that we consider here are the weighted caching problem, generalized caching, the set-cover problem, several graph optimization problems, routing, load balancing, and the problem of allocating ad-auctions. We also show that classic online problems such as the ski rental problem and the dynamic TCP-acknowledgement problem can be solved optimally using a simple primal–dual approach.

The primal–dual method has several advantages over existing methods. First, it provides a general recipe for the design and analysis of online algorithms. The linear programming formulation helps detecting the difficulties of the online problem, and the analysis of the competitive ratio is direct, without a potential function appearing "out of nowhere." Finally, since the analysis is done via duality, the competitiveness of the online algorithm is with respect to an optimal fractional solution, which can be advantageous in certain scenarios.

1

Introduction

The primal–dual method is a powerful algorithmic technique that has proved to be extremely useful for a wide variety of problems in the area of approximation algorithms. The method has its origins in the realm of exact algorithms, e.g., for matching and network flow. In the area of approximation algorithms, the primal–dual method has emerged as an important unifying design methodology starting from the seminal work of Goemans and Williamson [60].

Our goal in this survey is to extend the primal–dual method to the setting of online algorithms, and show that it is applicable to a wide variety of problems. The approach we propose has several advantages over existing methods:

- A general recipe for the design and analysis of online algorithms is developed.
- The framework is shown to be applicable to a wide range of fundamental online problems.
- A linear programming formulation helps detecting the difficulties of the online problem in hand.
- The competitive ratio analysis is direct, without a potential function appearing "out of nowhere."

- The competitiveness of the online algorithm is with respect to an optimal fractional solution.

In Section 2, we briefly provide the necessary background needed for the rest of the discussion. This includes a short exposition on linear programming, duality, offline approximation methods, and basic definitions of online computation. Many readers may already be familiar with these basic definitions and techniques; however, we advise the readers not to skip this chapter, and in particular the part on approximation algorithms. Techniques pertinent to approximation algorithms are presented in a way that allows the reader to later see the similarity to the online techniques we develop. This section also provides some of the basic notation that we use in the sequel. Section 3 gives a first taste of the primal–dual approach in the context of online algorithms via the well-understood ski rental problem. We show an alternative way of deriving optimal algorithms for the ski rental problem using a simple primal–dual approach. In Section 4, we lay the foundations for the online primal–dual approach and design the basic algorithms for the packing–covering framework. We also study two toy examples that demonstrate the online framework. The rest of the sections show how to apply the primal–dual approach to many interesting and fundamental problems. We tried to make the chapters independent of each other; however, there are still certain connections between chapters, and thus closely related problems appear in consecutive chapters and typically in increasing order of complexity.

Among the problems that we consider are the weighted caching problem, generalized caching, the online set-cover problem, several graph optimization problems, routing, load balancing, and even the problem of allocating ad-auctions. We also show that classic online problems like the dynamic TCP-acknowledgement problem can be optimally solved using a primal–dual approach. There are also several more problems that can be solved via the primal–dual approach and are not discussed here. Such problems are, for example, the admission control problem [5], the parking permit problem [83] and the inventory problem [31].

2

Necessary Background

In this section, we briefly overview the background needed for reading the rest of the survey. In Section 2.1, we discuss linear programming and duality. In Section 2.2, we discuss several classical methods for deriving (offline) approximation algorithms for intractable optimization problems. We demonstrate these ideas on the set-cover problem which we later consider in the online setting. In Section 2.3, we provide basic concepts and definitions related to online computation. This section is not meant to give a comprehensive introduction, but rather only provide the basic notation and definitions used later on in the text. For a more comprehensive discussion of these subjects, we refer the reader to the many excellent textbooks on these subjects. For more information on linear programming and duality, we refer the reader to [42]. For further information on approximation techniques we refer the reader to [90]. Finally, for more details on online computation and competitive analysis we refer the reader to [28].

2.1 Linear Programming and Duality

Linear programming is the problem of minimizing or maximizing a linear objective function over a feasible set defined by a set of linear

inequalities. There are several equivalent ways of formulating a linear program. For our purposes, the most convenient one is the following:

$$(P) : \min \sum_{i=1}^{n} c_i x_i \text{ s.t.}$$

For any $1 \leq j \leq m$:

$$\sum_{i=1}^{n} a_{ij} x_i \geq b_j, \quad \forall 1 \leq i \leq n, \quad x_i \geq 0.$$

It is well known that any linear program can be formulated in this way. We refer to such a minimization problem as the primal problem (P). Any vector $x = (x_1, x_2, \ldots, x_n)$ that satisfies all the constraints of (P) is referred to as a feasible solution to the linear program (P). Every primal linear program has a corresponding dual program. The dual linear program of (P) is a maximization linear program: it has m dual variables that correspond to the primal constraints and it has n constraints that correspond to the primal variables. The dual program (D) that corresponds to the linear program formulation (P) is the following:

$$(D) : \max \sum_{j=1}^{m} b_j y_j \text{ s.t.}$$

For any $1 \leq i \leq n$:

$$\sum_{j=1}^{m} a_{ij} y_j \leq c_i, \quad \forall 1 \leq j \leq m, \quad y_j \geq 0.$$

The useful properties of the dual program are summarized in the following theorems.

Theorem 2.1 (Weak duality). Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_m)$ be feasible solutions to the primal and the dual linear programs, respectively, then:

$$\sum_{i=1}^{n} c_i x_i \geq \sum_{j=1}^{m} b_j y_j.$$

Weak duality states that the value of any feasible dual solution is at most the value of any feasible primal solution. Thus, the dual program can actually be used as a lower bound for any feasible primal solution. The proof of this theorem is quite simple.

Proof.

$$\sum_{i=1}^{n} c_i x_i \geq \sum_{i=1}^{n} \left(\sum_{j=1}^{m} a_{ij} y_j \right) x_i \tag{2.1}$$

$$= \sum_{j=1}^{m} \left(\sum_{i=1}^{n} a_{ij} x_i \right) y_j \tag{2.2}$$

$$\geq \sum_{j=1}^{m} b_j y_j, \tag{2.3}$$

where inequality (2.1) follows since $y = (y_1, y_2, \ldots, y_m)$ is feasible and each x_i is non-negative. Equality (2.2) follows by changing the order of summation. Inequality (2.3) follows since $x = (x_1, x_2, \ldots, x_n)$ is feasible and each y_j is non-negative. □

The next theorem is sometimes referred to as the strong duality theorem. It states that if the primal and dual programs are bounded, then the optima of the two programs is equal. The proof of the strong duality theorem is harder and we only state the theorem here without a proof.

Theorem 2.2 (Strong duality). The primal linear program is feasible if and only if the dual linear program has a finite optimal solution. In this case, the value of the optimal solutions of the primal and dual programs is equal.

Finally, we prove an important theorem on approximate complementary slackness which is used extensively in the context of approximation algorithms.

Theorem 2.3 (Approximate complementary slackness). Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_m)$ be feasible solutions to the

primal and dual linear programs, respectively, satisfying the following conditions:

- *Primal complementary slackness*: For $\alpha \geq 1$, for each i, $1 \leq i \leq n$, if $x_i > 0$ then $c_i/\alpha \leq \sum_{j=1}^{m} a_{ij} y_i \leq c_i$.
- *Dual complementary slackness*: For $\beta \geq 1$, for each j, $1 \leq j \leq m$, if $y_j > 0$ then $b_j \leq \sum_{i=1}^{n} a_{ij} x_i \leq b_j \cdot \beta$.

Then

$$\sum_{i=1}^{n} c_i x_i \leq \alpha \cdot \beta \sum_{j=1}^{m} b_j y_j.$$

In particular, if the complementary slackness conditions hold with $\alpha = \beta = 1$, then we get that x and y are both optimal solutions to the primal and dual linear programs, respectively. The proof of the theorem is again very short.

Proof.

$$\sum_{i=1}^{n} c_i x_i \leq \alpha \sum_{i=1}^{n} \left(\sum_{j=1}^{m} a_{ij} y_i \right) x_i \tag{2.4}$$

$$= \alpha \sum_{j=1}^{m} \left(\sum_{i=1}^{n} a_{ij} x_i \right) y_j \tag{2.5}$$

$$\leq \alpha \cdot \beta \sum_{j=1}^{m} b_j y_j, \tag{2.6}$$

where (2.4) follows from the primal complementary slackness condition. Equality (2.5) follows by changing the order of summation, and Inequality (2.6) follows from the dual complementary slackness condition. □

Theorem 2.3 gives an efficient tool for proving that a solution is approximately optimal. Consider, for example, a minimization problem. Suppose you can find primal and dual solutions that satisfy the (approximate) complementary slackness conditions. Then, the solution for the

minimization problem is at most $\alpha \cdot \beta$ times a feasible dual solution. Since, by weak duality (Theorem 2.1), the value of any dual solution is a lower bound on the value of any primal solution, the solution obtained is also at most $\alpha \cdot \beta$ times the optimal primal solution.

Covering/packing linear formulations: A special class of linear programs are those in which the coefficients a_{ij}, b_j and c_i are all non-negative. In this case, the primal program forms a *covering problem* and the dual program forms a *packing problem*. The meaning of these names will become clear in Section 2.2 where we discuss the set cover problem. In the sequel, we sometimes use the notion of *covering–packing* primal–dual pair.

2.2 Approximation Algorithms

In this section, we give a very short background on some basic methods for developing approximation algorithms for intractable problems. Later on we show that similar ideas can be extended and used in the context of online algorithms. We start by formally defining the notions of optimization problems and approximation factors. In an (offline) minimization optimization problem we are given set of instances $\mathbb{I}$. For each instance $I \in \mathbb{I}$ there is a set of feasible solutions. Each feasible solution is associated with a *cost*. Let $\mathrm{OPT}(I)$ be the cost of the minimal feasible solution for instance I. A polynomial time algorithm A is called a c-approximation for a minimization optimization problem if for every instance I it outputs a solution with cost at most $c \cdot \mathrm{OPT}(I)$. The definitions for maximization optimization problems are analogous. In this case, each instance is associated with a *profit*. A c-approximation algorithm is guaranteed to return a solution with cost at least $\mathrm{OPT}(I)/c$, where $\mathrm{OPT}(I)$ is the maximum profit solution.

The set-cover problem: We demonstrate several classic ideas used for developing approximation algorithms via the set cover problem. In the set-cover problem, we are given a set of n elements $X = e_1, e_2, \ldots, e_n$, and a family $\mathbb{S} = s_1, s_2, \ldots, s_m$ of subsets of X, $|\mathbb{S}| = m$. Each set s_j is associated with a non-negative cost c_s. A *cover* is a collection of sets such that their union is X. The objective is to find a cover of

X of minimum cost. This problem is known to be NP-hard. Linear programming relaxations constitute a very useful way for obtaining lower bounds on the optimum value of a given combinatorial problem. To this end, we introduce a non-negative variable x_s for each set $s \in \mathbb{S}$. Initially, we formulate the problem as an integer program, allowing x_s to be either 0 or 1. The following is an integer formulation of the set-cover problem:

$$\min \sum_{s \in \mathbb{S}} c_s x_s \text{ s.t.}$$

For each element e_i $(1 \leq i \leq n)$:

$$\sum_{s|e_i \in S} x_s \geq 1, \quad \forall s \in \mathbb{S}, \quad x_s \in \{0,1\}.$$

It is not hard to verify that the set of feasible solution to this program corresponds to the set of feasible covers. Thus, a minimum cover corresponds to an optimal solution to the integer program. Next, we relax the constraint $x_s \in \{0,1\}$ and get the following linear formulation of the problem:

$$(P) : \min \sum_{s \in \mathbb{S}} c_s x_s \text{ s.t.}$$

For each element e_i $(1 \leq i \leq n)$:

$$\sum_{s|e_i \in S} x_s \geq 1, \quad \forall s \in \mathbb{S}, \quad x_s \geq 0.$$

Since the feasible space of the linear formulation contains all the integral solutions of the integer formulation, we get that the optimal solution to the linear formulation is a lower bound on the value of any integral set cover solution. We also remark that there is no need to demand that $x_s \leq 1$, since a minimum cost solution will never increase the value of any x_s above 1. A solution to the linear formulation is called a fractional set cover solution. In such a relaxed set cover solution, one is allowed to take a fraction x_s of each set and pay only $c_s x_s$ for this fraction. The restriction is that the sum of fractions of sets that contain each element e_i should be at least 1. In the corresponding dual

linear program of (P) there is a variable for each element e_i. The dual program (D) is the following:

$$(D) : \max \sum_{e \in X} y_{e_i} \text{ s.t.}$$

For each set $s \in \mathbb{S}$:

$$\sum_{e_i \in S} y_{e_i} \leq c_s, \quad \forall 1 \leq i \leq n, \quad y_{e_i} \geq 0.$$

We next demonstrate several classic approximation techniques using the set cover problem as our running example.

2.2.1 Dual Fitting

We present the dual fitting method by analyzing the greedy algorithm for the set cover problem. The greedy algorithm is depicted below.

Greedy algorithm: Initially, $\mathbb{C} = \emptyset$. Let U be the set of yet uncovered elements.
As long as $U \neq \emptyset$, let $s \in \mathbb{S}$ be the set that minimizes the ratio $c_s/|U \cap s|$:

(1) Add s to $\mathbb{C}$ and set $x_s \leftarrow 1$.
(2) For each $e_i \in (U \cap s)$, $y_{e_i} \leftarrow c_s/|U \cap s|$.

Theorem 2.4. The greedy algorithm is an $O(\log n)$-approximation algorithm for the set-cover problem.

Proof. We are going to show that the algorithm produces throughout its execution both primal and dual solutions. Let P and D be the values of the objective functions of the primal and dual solutions produced, respectively. Initially, $P = D = 0$. We focus on a single iteration of the algorithm and denote by ΔP and ΔD the change in the primal and dual cost, respectively. We prove three simple claims:

(1) The algorithm produces a primal (covering) feasible solution.
(2) In each iteration: $\Delta P \leq \Delta D$.

(3) Each packing constraint in the dual program is violated by a multiplicative factor of at most $O(\log n)$.

The proof follows immediately from the three claims together with weak duality. First, by claim (1) our solution is feasible. By the fact that initially $P = D = 0$ and by claim (2) we get that we produce primal and dual solutions such that $P \leq D$. Finally, by claim (3) we get that dividing the dual solution by $c \log n$, for a constant c, yields a dual feasible solution with value $D' = D/(c \log n)$. Therefore, we get that the primal cost is at most $c \log n$ times a feasible dual solution. Since by weak duality a feasible dual solution is at most the cost of any primal solution, we get that the primal solution is at most $c \log n$ times the optimal (minimal) primal solution.

Proof of (1): Clearly, if there exists a feasible primal solution, then the algorithm will also produce a feasible solution.

Proof of (2): Consider an iteration of the algorithm in which U is the set of uncovered elements and set s is chosen. The change in the primal cost (due to the addition of set s) is c_s. In the dual program, we set the variables corresponding to the $|U \cap s|$ elements covered in the iteration to $c_s/|U \cap s|$. Thus, the total change in the dual profit is also c_s. Note that we only set the dual variable corresponding to each element only once. This happens in the iteration in which the element is covered.

Proof of (3): Consider the dual constraint corresponding to set s. Let $e_1, e_2, \ldots, e_k$ be the elements belonging to set s, sorted with respect to the order they were covered by the greedy algorithm. Consider element e_i; we claim that $y_{e_i} \leq c_s/(k - i + 1)$. This is true since the algorithm chooses the set that minimizes the ratio of cost to number of uncovered elements, which is in fact the value assigned to y_{e_i}. At the time e_i is covered by the greedy algorithm, set s contained at least $k - i + 1$ elements that were still uncovered, thus we get that $y_{e_i} \leq c_s/(k - i + 1)$. Therefore, we get for set s:

$$\sum_{e_i \in s} y_{e_i} \leq \sum_{i=1}^{k} \frac{c_s}{k - i + 1} = H_k \cdot c_s = c_s \cdot O(\log n),$$

where H_k is the kth harmonic number. □

2.2.2 Randomized Rounding

In this section, we design a different $O(\log n)$-approximation algorithm for the set cover problem using a useful technique called *randomized rounding*. The first step is to compute an optimal fractional solution to linear program (P). Then, the fractional solution is rounded by treating the fractions as probabilities. We describe the rounding algorithm slightly different from standard textbooks; however, this description will be useful in the sequel. The rounding algorithm is the following:

Randomized rounding algorithm:

(1) For each set $s \in \mathbb{S}$, choose $2 \ln n$ independently random variables $X(s,i)$ uniformly at random in the interval $[0,1]$.
(2) For each set s, let $\Theta(s) = \min_{i=1}^{2 \ln n} X(s,i)$.
(3) Solve linear program (P).
(4) Take set s to the cover if $\Theta(s) \leq x_s$.

Theorem 2.5. The algorithm produces a solution with the following properties:

(1) The expected cost of the solution is $O(\log n)$ times the cost of the fractional solution.
(2) The solution is feasible with probability $1 - 1/n > 1/2$.

Since the fractional solution provides a lower bound on any integral solution, we get that the algorithm is an $O(\log n)$-approximation.

Proof. The proof strongly uses the fact that in a feasible solution, for each element, the sum of fractions of the sets containing it is at least 1.

To prove (1) note that for each i, $1 \leq i \leq 2 \ln n$, the probability that $X(s,i) \leq x_s$ is exactly x_s. The probability that s is chosen to the solution is the probability that there exists an i, $1 \leq i \leq 2 \ln n$, such that $X(s,i) \leq x_s$. Let A_i be the event that $X(s,i) \leq x_s$. Thus, the probability that s is chosen to the solution is the probability of $\bigcup_{i=1}^{2 \ln n} A_i$. By the union bound this probability is at most the sum of

the probabilities of the different events, which is $2x_s \ln n$. Therefore, using linearity of expectation the expected cost of the solution is at most $2 \ln n$ times the cost of the fractional solution.

To prove (2) pick an element e. Fix any i, $1 \leq i \leq 2 \ln n$. The probability that e is not covered due to i is the probability that none of the variables $X(s,i)$ (for all sets s, $e \in s$) result in choosing a set s covering e. This probability is

$$\prod_{s \in \mathbb{S} | e \in s} (1 - x_s) \leq \exp\left(- \sum_{s \in \mathbb{S} | e \in s} x_s \right) \leq \exp(-1),$$

where the first inequality follows since $1 - x \leq \exp(-x)$. The second inequality follows since each element is fractionally covered. Since we choose $2 \ln n$ random variables independently, the probability that e is not covered is at most $\exp(-2 \ln n) = 1/n^2$. Using the union bounds we get that the probability that there exists an element e which is not covered is at most $n \cdot 1/n^2 = 1/n$. □

2.2.3 The Primal–Dual Schema

In this section, we give a third approximation algorithm for the set cover problem which is based on the primal–dual schema. The algorithm is the following:

Primal–dual algorithm:
While there exists an uncovered element e_i:

(1) Increase the dual variable y_{e_i} continuously.
(2) If there exists a set s such that $\sum_{e \in s} y_e = c_s$: take s to the cover and set $x_s \leftarrow 1$.

Let f be the maximum frequency of an element (i.e., the maximum number of sets an element belongs to).

Theorem 2.6. The algorithm is an f-approximation for the set-cover problem.

Proof. Clearly, the primal solution produced by the algorithm is feasible, since we pick sets to the cover as long as the solution is infeasible. The dual solution produced by the algorithm is also feasible, since set s is taken to the solution only when the dual constraint corresponding to it becomes tight. From then on, we never increase a dual variable corresponding to an element belonging to s. Finally, the primal complementary slackness condition holds with $\alpha = 1$, since, if $x_s > 0$, then $\sum_{e \in s} y_{e_i} = c_s$. The dual complementary slackness condition holds with $\beta = f$, since, if $y_{e_i} > 0$, then $1 \leq \sum_{s|e \in s} x_s \leq f$. Thus, by Theorem 2.3, we get that the algorithm is an f-approximation. □

Remark 2.7. For the special case of the vertex cover problem this algorithm is a 2-approximation.

2.3 Online Computation

The traditional design and analysis of algorithms assumes that complete knowledge of the entire input is available to an algorithm. However, this is not the case in an online problem, where the input is revealed in parts, and an online algorithm is required to respond to each new input upon arrival. Previous decisions of the online algorithm cannot be revoked. Thus, the main issue in online computation is obtaining good performance in the face of uncertainty, since the "future" is unknown to the algorithm. A standard measure by now for evaluating the performance of an online algorithm is the *competitive ratio*, which compares the performance of an online algorithm to that of an offline algorithm which is given the whole input sequence beforehand.

The precise definition of the competitive factor is the following. Suppose we are given a minimization optimization problem I. For each instance of I there is a set of feasible solutions. Each feasible solution is associated with a *cost*. Let $\mathrm{OPT}(I)$ be the optimal cost of a feasible solution for instance I. In the online case, the instance is given to the algorithm in parts. An online algorithm A is said to be c-competitive for I if for every instance of I it outputs a solution of

cost at most $c \cdot \mathrm{OPT}(I) + \alpha$, where the additive term α is independent of the request sequence. if $\alpha = 0$ then the algorithm is called *strictly* c-competitive. We are not going to distinguish between the two notions. The definition of competitiveness of maximization optimization problems is analogous. When considering a maximization problem each instance is associated with a *profit*. A c-competitive algorithm is guaranteed to return a solution with cost at least $\mathrm{OPT}(I)/c - \alpha$, where $\mathrm{OPT}(I)$ is the maximum profit solution, and α is an additive term independent of the request sequence.

A common concept in competitive analysis is that of an *adversary*. The online solution can be viewed as a game between an online algorithm and a malicious adversary. While the online algorithm's strategy is to minimize its cost, the adversary's strategy is to construct the worst possible input for the algorithm. Using this view, the adversary produces a sequence $\sigma = \sigma_1, \sigma_2, \ldots$ of requests that define the instance I. A c-competitive online algorithm should then be able to produce a solution of cost no more than c times $\mathrm{OPT}(\sigma)$ for every request sequence σ.

There are several known natural models of adversaries in the context of randomized online algorithms. In this work, we only consider a model in which the adversary knows the online algorithm, as well as the probability distribution used by the online algorithm to make its random decisions. However, the adversary is unaware of the actual random choices made by the algorithm throughout its execution. This kind of adversary is called an *oblivious adversary*. A randomized online algorithm is c-competitive against an oblivious adversary, if for every request sequence σ, the expected cost of the algorithm on σ is at most $c \cdot \mathrm{OPT}(\sigma) + \alpha$. The expectation is taken over all random choices made by the algorithm. Since the oblivious adversary has no information about the actual random choices made by the algorithm, the sequence σ can be constructed ahead of time and $\mathrm{OPT}(\sigma)$ is not a random variable. In the sequel, whenever we have a randomized online algorithm, we simply say that it is c-competitive, and mean that it is c-competitive against an oblivious adversary. Again, the definitions for maximization problems are analogous.

2.4 Notes

The dual-fitting analysis in Section 2.2.1 of the greedy heuristic is due to [78, 41]. The algorithm in Section 2.2.3 is due to Bar-Yehuda and Even [17]. The set-cover problem is a classic NP-hard problem that was studied extensively in the literature. The best approximation factor achievable for it in polynomial time (assuming $P \neq \text{NP}$) is $\Theta(\log n)$ [41, 47, 68, 78].

The introduction here is only meant to establish basic notation and terminology for the rest of our discussion. The area of linear programming, duality, approximation algorithms, and online computation have been studied extensively. For more information on linear programming and duality we refer the reader to [42]. For further information on approximation techniques we refer the reader to [90]. Finally, for more details on online computation and competitive analysis we refer the reader to [28].

3

A First Glimpse: The Ski Rental Problem

Let us start with a classic online problem, *ski rental*, through which we will demonstrate the basic ideas underlying the online primal–dual approach. At a ski resort renting skis costs \$1 per day, while buying skis costs $\$B$. A skier arrives at the ski resort for a ski vacation and has to decide whether to rent or buy skis. However, an unknown factor is the number of remaining skiing days that are left before the snow melts. (We should note that this is the skier's last vacation.) In spite of its apparent simplicity, the ski rental problem captures the essence of online rent or buy dilemmas. The ski rental problem is well understood. There exists a simple deterministic 2-competitive algorithm for the problem and a randomized $e/(e-1)$-competitive algorithm. Both results are tight. We show here how to obtain these results using a primal–dual approach.

The first step towards obtaining an online primal–dual algorithm is formulating the problem in hand as a linear program. Since the offline ski rental problem is so simple, casting it as a linear program may seem a bit unnatural. However, this formulation turns out to be very useful. We define an indicator variable x which is set to 1 if the skier buys the skis. For each day j, $1 \leq j \leq k$, we define an indicator variable z_j which

is set to 1 if the skier decides to rent skis on day j. The constraints guarantee that on each day we either rent skis or buy them. This gives us the following integer formulation for the ski rental problem:

$$\min B \cdot x + \sum_{j=1}^{k} z_j$$

For each day j:

$$x + z_j \geq 1, \quad x \in \{0,1\}, \ \forall j, z_j \in \{0,1\}.$$

We next relax the problem and allow x and each z_j to be in $[0,1]$. The linear program is given in Figure 3.1 (the primal program). Note that there is always an optimal integral solution, and thus the relaxation has no integrality gap. The dual program is also extremely simple and has a variable y_j corresponding to each day j. We then have a constraint $y_j \leq 1$ that corresponds to each primal variable z_j, and a constraint $\sum_{j=1}^{k} y_j \leq B$ that corresponds to the primal variable x. Note that the linear programming formulation forms a covering–packing primal–dual pair.

Next, consider the online scenario in which k (the number of ski days) is unknown in advance. This scenario can be captured in the linear formulation in a very natural way. Whenever we have a new ski day, the primal linear program is updated by adding a new covering constraint. The dual program is updated by adding a new dual variable which is added to the packing constraints. The online requirement is that previous decisions cannot be undone. That is, if we already rented skis yesterday, we cannot change this decision today. This requirement is captured in the primal linear program by the restriction that the primal variables are monotonically non-decreasing over time.

Dual (Packing)		Primal (Covering)	
Maximize:	$\sum_{j=1}^{k} y_j$	Minimize :	$B \cdot x + \sum_{j=1}^{k} z_j$
Subject to:		Subject to:	
	$\sum_{j=1}^{k} y_j \leq B$	For each day j:	$x + z_j \geq 1$
For each day j:	$0 \leq y_j \leq 1$		$x \geq 0, \forall j, z_j \geq 0$

Fig. 3.1 The fractional ski problem (the primal) and the corresponding dual problem.

Obtaining an optimal deterministic online algorithm for the ski rental problem is very simple. First, rent the skis for the first $B-1$ days, and then buy them on the Bth day. If $k < B$, the algorithm is optimal. If $k \geq B$, then the algorithm spends $\$2B$ dollars, while the optimal solution buys skis on the first day and spends only $\$B$ dollars. Thus, the algorithm is 2-competitive. This simple algorithm and analysis can be obtained in a somewhat less natural way using a primal–dual analysis. On the jth day, a new primal constraint $x + z_j \geq 1$ and a new dual variable y_j arrive. If the primal constraint is already satisfied, then do nothing. Otherwise, increase y_j continuously until some dual constraint becomes tight. Set the corresponding primal variable to be 1. The above algorithm is a simple application of the primal–dual schema, and its analysis is very simple using the approximate complementary slackness conditions: If $y_j > 0$ then $1 \leq x + z_j \leq 2$. Moreover, if $x > 0$ then $\sum_{j=1}^{k} y_j = B$, and if $z_j > 0$ then $y_j = 1$. Thus, by Theorem 2.3, the algorithm is 2-competitive. It is not hard to see that both of the above algorithms are actually equivalent.

Developing an optimal randomized algorithm is not as straightforward as the deterministic one, yet the primal–dual approach turns out to be very useful towards this end. The first step is designing a deterministic fractional competitive algorithm. Recall that in the fractional case the primal variables can be in the interval $[0,1]$, and the variables are required to be monotonically non-decreasing over time, during the execution of the algorithm. The online algorithm is the following:

(1) Initially, $x \leftarrow 0$.
(2) Each new day (jth new constraint), if $x < 1$:
    (a) $z_j \leftarrow 1 - x$.
    (b) $x \leftarrow x(1 + 1/B) + 1/(c \cdot B)$. (The value of c will be determined later.)
    (c) $y_j \leftarrow 1$.

The analysis is simple. We show that:

(1) The primal and dual solutions are feasible.
(2) In each iteration (day), the ratio between the change in the primal and dual objective functions is bounded by $(1 + 1/c)$.

Using the weak duality theorem (Theorem 2.1) we immediately conclude that the algorithm is $(1 + 1/c)$-competitive.

The proof is very simple. Denote the number of ski days by k. First, since we set $z_j = 1 - x$ (whenever $x < 1$), the primal solution produced is feasible. To show feasibility of the dual solution, we need to show that

$$\sum_{j=1}^{k} y_j \leq B.$$

We prove this by showing that $x \geq 1$ after at most B days of ski. Denote the increments of x (in each day) by $x_1, x_2, \ldots, x_k$, where $x = \sum_{j=1}^{k} x_j$. It can be easily seen that $x_1, x_2, \ldots, x_k$ is a geometric sequence, defined by $x_1 = 1/(cB)$ and $q = 1 + 1/B$. Thus, after B days,

$$x = \frac{\left(1 + \frac{1}{B}\right)^B - 1}{c}.$$

Setting $c = (1 + 1/B)^B - 1$ guarantees that $x = 1$ after B days.

Second, if $x < 1$, in each iteration the dual objective function increases by 1, and the increase in the primal objective function is $B\Delta x + z_j = x + 1/c + 1 - x = 1 + 1/c$, and thus the ratio is $(1 + 1/c)$. Concluding, the competitive ratio is $1 + 1/c \approx e/(e - 1)$ for $B \gg 1$.

Converting the fractional solution into a randomized competitive algorithm with the same competitive ratio is easy. We arrange the increments of x on the $[0,1]$ interval and choose ahead of time (before executing the algorithm) $\alpha \in [0,1]$ uniformly in random. We are going to buy skis on the day corresponding to the increment of x to which α belongs.

We now analyze the expected cost of the randomized algorithm. Since the probability of buying skis on the jth day is equal to x_j, the expected cost of buying skis is precisely $B \cdot \sum_{j=1}^{k} x_j = Bx$, which is exactly the first term in the primal objective function. The probability of renting skis on the jth day is equal to the probability of *not* buying skis on or before the jth day, which is $1 - \sum_{i=1}^{j} x_i$. Since

$$z_j = 1 - \sum_{i=1}^{j-1} x_i \geq 1 - \sum_{i=1}^{j} x_i,$$

we get that the probability of renting on the jth day is at most z_j, corresponding to the second term in the primal objective function. Thus, by linearity of expectation, for any number of ski days, the expected cost of the randomized algorithm is at most the cost of the fractional solution.

3.1 Notes

The results in this chapter are based on the work of Buchbinder et al. [30] who showed how to obtain a randomized $e/(e-1)$-competitive algorithm via the primal–dual approach. The randomized $e/(e-1)$-competitive factor for the ski rental problem was originally obtained by Karlin et al. [71]. The role of randomization in the ski problem was later restudied, along with other problems, in [70]. The deterministic 2-competitive algorithm is due to [72].

4

The Basic Approach

We are now ready to extend the ideas used in the previous chapter for the ski rental problem and develop a general recipe for online problems which can be formulated as packing–covering linear programs. In Section 4.1, we formally define a general online framework for packing–covering problems. In Section 4.2, we develop several competitive online algorithms for this framework. In Section 4.3, we prove lower bounds and show that these algorithms are optimal. Finally, in Section 4.4, we give two simple examples that utilize the new ideas developed here.

4.1 The Online Packing–Covering Framework

We are going to define here a general online framework for packing–covering problems. Let us first consider an "offline" covering linear problem. The objective is to minimize the total cost given by a linear cost function $\sum_{i=1}^{n} c_i x_i$. The feasible solution space is defined by a set of m linear constraints of the form $\sum_{i=1}^{n} a(i,j)x_i \geq b(j)$, where the entries $a(i,j), b(j)$ and c_i are *non-negative*. For simplicity, we consider in this chapter a simpler setting in which $b(j) = 1$ and $a(i,j) \in \{0,1\}$. In Section 14, we show how to extend the ideas we present here to handle

general (non-negative) values of $a(i,j)$ and $b(j)$. In the simpler setting, each covering constraint j can be associated with a set $S(j)$ such that $i \in S(j)$ if $a(i,j) = 1$. The jth covering constraint then reduces to simply $\sum_{i \in S(j)} x_i \geq 1$. Any primal covering instance has a corresponding dual packing problem that provides a lower bound on any feasible solution to the instance. A general form of a (simpler) primal covering problem along with its dual packing problem is given in Figure 4.1.

The *online covering problem* is an online version of the covering problem. In the online setting, the cost function is known in advance, but the linear constraints that define the feasible solution space are given to the algorithm one-by-one. In order to maintain a feasible solution to the current set of given constraints, the algorithm is allowed to increase the variables. It may not, however, decrease any previously increased variable. The objective of the algorithm is to minimize the objective function. The reader may already have noticed that this online setting captures the ski rental problem (Chapter 3) as a special case.

The *online packing problem* is an online version of the packing problem. In the online setting, the values c_i $(1 \leq i \leq n)$ are known in advance. However, the profit function and the exact packing constraints are not known in advance. In the jth round, a new variable y_j is introduced, along with the set of packing constraints it appears in. The algorithm may increase the value of a variable y_j only in the round in which it is given, and may not decrease or increase the values of any previously given variables. Note that variables that have not yet been introduced may also later appear in the same packing constraints. This actually means that each packing constraint is revealed to the algorithm gradually. The objective of the algorithm is to maximize the objective function while maintaining the feasibility of all packing constraints. Although this online setting seems at first glance a bit

(P): Primal (Covering)		(D): Dual (Packing)	
Minimize:	$\sum_{i=1}^{n} c_i x_i$	Maximize:	$\sum_{j=1}^{m} y_j$
subject to:		subject to:	
$\forall 1 \leq j \leq m$:	$\sum_{i \in S(j)} x_i \geq 1$	$\forall 1 \leq i \leq n$:	$\sum_{j \mid i \in S(j)} y_j \leq c_i$
$\forall 1 \leq i \leq n$:	$x_i \geq 0$	$\forall 1 \leq j \leq m$:	$y_j \geq 0$

Fig. 4.1 Primal (covering) and dual (packing) problems.

unnatural, we later show that many natural online problems reduce to this online setting.

Clearly, at any point of time the linear constraints that have appeared so far define a *sub-instance* of the final covering instance. The dual packing problem of this sub-instance is a *sub-instance* of the final dual packing problem. In the dual packing sub-instance, only part of the dual variables are known, along with their corresponding coefficients. Thus, the two sub-instances derived from the above online settings form a *primal–dual pair*.

The algorithms we propose in the next section maintain at each step solutions for both the primal and dual sub-instances. When a new constraint appears in the online covering problem, our algorithms also consider the new *corresponding* dual variable and its coefficients. Conversely, when a new variable along with its coefficients appear in the online fractional packing problem, our algorithms also consider the *corresponding* new constraint in the primal sub-instance.

4.2 Three Simple Algorithms

In this section, we present three algorithms with the same (optimal) performance guarantees for the online covering/packing problem. Although the performance of all three algorithms in the worst case is the same, their properties do vary, and thus certain algorithms are better suited for particular applications. Also, the ideas underlying each of the algorithms can be extended later to more complex settings. The first algorithm is referred to as the *basic discrete algorithm* and is a direct extension of the algorithm for the ski rental in Section 3. The algorithm is the following:

Algorithm 1:
Upon arrival of a new primal constraint $\sum_{i \in S(j)} x_i \geq 1$ and the corresponding dual variable y_j:

(1) While $\sum_{i \in S(j)} x_i < 1$:
  (a) For each $i \in S(j)$: $x_i \leftarrow x_i(1 + 1/c_i) + 1/(|S(j)|c_i)$.
  (b) $y_j \leftarrow y_j + 1$.

We assume that each $c_i \geq 1$. This assumption is not restrictive and we discuss the reason for that later on. Let $d = \max_j |S(j)| \leq m$ be the maximum "size" of a covering constraint. We prove the following theorem:

Theorem 4.1. The algorithm produces:

- A (fractional) covering solution which is $O(\log d)$-competitive.
- An "integral" packing solution which is 2-competitive and violates each packing constraint by at most a factor of $O(\log d)$.

We remark that it is possible to obtain a feasible packing solution by scaling the update of each y_j by a factor of $O(\log\ d)$. This, however, will yield a non-integral packing solution. It is also beneficial for the reader to note the similarity ("in spirit") between the proof of Theorem 4.1 and the dual fitting based proof of the greedy heuristic for the set cover problem in Section 2.2.1.

Proof. Let P and D be the values of the objective function of the primal and the dual solutions the algorithm produces, respectively. Initially, $P = D = 0$. The dual variables start from zero and increase in increments of one unit, the therefore the dual solution is integral.

Let ΔP and ΔD be the changes in the primal and dual cost, respectively, in a particular iteration of the algorithm in which we execute the inner loop. We prove three simple claims:

(1) The algorithm produces a primal (covering) feasible solution.
(2) In each iteration: $\Delta P \leq 2\Delta D$.
(3) Each packing constraint in the dual program is violated by at most $O(\log d)$.

The proof then follows immediately from the three claims together with weak duality.

Proof of (1): Consider a primal constraint $\sum_{i \in S(j)} x_i \geq 1$. During the jth iteration the algorithm increases the values of the variables x_i until

the constraint is satisfied. Subsequent increases of the variables cannot result in infeasiblity.

Proof of (2): Whenever the algorithm updates the primal and dual solutions, the change in the dual profit is 1. The change in the primal cost is

$$\sum_{i \in S(j)} c_i \Delta x_i = \sum_{i \in S(j)} c_i \left(\frac{x_i}{c_i} + \frac{1}{|S(j)|c_i} \right) = \sum_{i \in S(j)} \left(x_i + \frac{1}{|S(j)|} \right) \leq 2,$$

where the final inequality follows since the covering constraint is infeasible at the time of the update.

Proof of (3): Consider any dual constraint $\sum_{j|i \in S(j)} y_j \leq c_i$. Whenever we increase a variable y_j, $i \in S(j)$, by one unit we also increase the variable x_i in line (1a). We prove by simple induction that the variable x_i is bounded from below by the sum of a geometric sequence with $a_1 = 1/(dc_i)$ and $q = (1 + 1/c_i)$. That is,

$$x_i \geq \frac{1}{d} \left(\left(1 + \frac{1}{c_i} \right)^{\sum_{j|i \in S(j)} y_j} - 1 \right). \tag{4.1}$$

Initially, $x_i = 0$, so the induction hypothesis holds. Next, consider an iteration in which variable y_k increases by 1. Let $x_i(\text{start})$ and $x_i(\text{end})$ be the values of variable x_i before and after the increment, respectively. Then,

$$\begin{aligned}
x_i(\text{end}) &= x_i(\text{start}) \left(1 + \frac{1}{c_i} \right) + \frac{1}{|S(j)|c_i} \\
&\geq x_i(\text{start}) \left(1 + \frac{1}{c_i} \right) + \frac{1}{d \cdot c_i} \\
&\geq \frac{1}{d} \left(\left(1 + \frac{1}{c_i} \right)^{\sum_{j|i \in S(j) \setminus \{k\}} y_j} - 1 \right) \left(1 + \frac{1}{c_i} \right)^{y_k} + \frac{1}{d \cdot c_i} \\
&= \frac{1}{d} \left(\left(1 + \frac{1}{c_i} \right)^{\sum_{j|i \in S(j)} y_j} - 1 \right).
\end{aligned}$$

Note that the inductive hypothesis is applied to $x_i(\text{start})$.

Next, observe that the algorithm never updates any variable $x_i \geq 1$ (since it cannot belong to any unsatisfied constraint). Since each $c_i \geq 1$

and $d \geq 1$, we have that $x_i < 1(1+1)+1 = 3$. Together with inequality (4.1) we get that:

$$3 \geq x_i \geq \frac{1}{d}\left(\left(1+\frac{1}{c_i}\right)^{\sum_{j|i\in S(j)} y_j} - 1\right).$$

Using again the fact that $c_i \geq 1$ and simplifying we get the desired result:

$$\sum_{j|i\in S(j)} y_j \leq c_i \log_2(3d+1) = c_i \cdot O(\log d).$$

□

The basic discrete algorithm is extremely simple and we show in the sequel its many applications. We are now going to derive a slightly different algorithm, which has a continuous flavor and is more in the spirit of the primal–dual schema. This algorithm will also be of guidance for gaining intuition about the right relationship between primal and dual variables. The algorithm is the following:

Algorithm 2:
Upon arrival of a new primal constraint $\sum_{i\in S(j)} x_i \geq 1$ and the corresponding dual variable y_j:

(1) While $\sum_{i\in S(j)} x_i < 1$:
   (a) Increase the variable y_j continuously.
   (b) For each variable x_i that appears in the (yet unsatisfied) primal constraint increase x_i according to the following function:

$$x_i \leftarrow \frac{1}{d}\left[\exp\left(\frac{\ln(1+d)}{c_i}\sum_{j|i\in S(j)} y_j\right) - 1\right].$$

Note that the exponential function for variable x_i contains dual variables that correspond to future constraints. However, these variables are all initialized to 0, so they do not contribute to the value of the

function. Although the algorithm is described in a continuous fashion, it is not hard to implement it in a discrete fashion in any desired accuracy. We discuss the intuition of the exponential function we use after proving the following theorem:

Theorem 4.2. The algorithm produces:

- A (fractional) covering solution which is feasible and $O(\log d)$-competitive.
- A (fractional) packing solution which is feasible and $O(\log d)$-competitive.

Proof. Let P and D be the values of the objective function of the primal and the dual solutions produced by the algorithm, respectively. Initially, $P = D = 0$. We prove three simple claims:

(1) The algorithm produces a primal (covering) feasible solution.
(2) In each iteration j: $\partial P/\partial y_j \leq 2\ln(1+d) \cdot \partial D/\partial y_j$.
(3) Each packing constraint in the dual program is feasible.

The theorem then follows immediately from the three claims together with weak duality.

Proof of (1): Consider a primal constraint $\sum_{i \in S(j)} x_i \geq 1$. During the iteration in which the the jth primal constraint and dual variable y_j appear, the algorithm increases the values of the variables x_i until the constraint is satisfied. Subsequent increases of variables cannot result in infeasibility.

Proof of (2): Whenever the algorithm updates the primal and dual solutions, $\partial D/\partial y_j = 1$. The derivative of the primal cost is

$$\begin{aligned}\frac{\partial P}{\partial y_j} &= \sum_{i \in S(j)} c_i \frac{\partial x_i}{\partial y_j} \\ &= \sum_{i \in S(j)} c_i \left(\frac{\ln(1+d)}{c_i} \frac{1}{d} \left[\exp\left(\frac{\ln(1+d)}{c_i} \sum_{j | i \in S(j)} y_j \right) \right] \right)\end{aligned}$$

$$= \ln(1+d) \sum_{i \in S(j)} \left(\frac{1}{d} \left[\exp\left(\frac{\ln(1+d)}{c_i} \sum_{j | i \in S(j)} y_j \right) - 1 \right] + \frac{1}{d} \right)$$

$$= \ln(1+d) \sum_{i \in S(j)} \left(x_i + \frac{1}{d} \right) \leq 2 \ln(1+d). \tag{4.2}$$

The last inequality follows since the covering constraint is infeasible.

Proof of (3): Consider any dual constraint $\sum_{j|i \in S(j)} y_j \leq c_i$. The corresponding variable x_i is always at most 1, since otherwise it cannot belong to any unsatisfied constraint. Thus, we get that:

$$x_i = \frac{1}{d} \left[\exp\left(\frac{\ln(1+d)}{c_i} \sum_{j | i \in S(j)} y_j \right) - 1 \right] \leq 1.$$

Simplifying we get that:

$$\sum_{j | i \in S(j)} y_j \leq c_i.$$

□

Discussion: As can be seen from the proof, the basic discrete algorithm and the continuous algorithm are essentially the same, since $(1 + 1/c_i)$ is approximately $\exp(1/c_i)$. The function in the continuous algorithm is then approximated by Inequality (4.1) in Theorem 4.1. The approximate equality is true as long as c_i is not too small, which is why the assumption that $c_i \geq 1$ is needed in the discrete algorithm. In addition, the discrete algorithm allows the primal variables to exceed the value of 1, which is unnecessary (and can easily be avoided). For these reasons, the proof of the continuous algorithm is a bit simpler. However, in contrast, the description of the discrete algorithm is simpler and more intuitive. Also, in the discrete algorithm, it is not necessary to know the value of d in advance (as long as it is not needed to scale down the dual variables to make the dual solution feasible).

The reader may wonder at this point how did we choose the function used in the algorithm for updating the primal and dual variables. We will try to give here a systematic way of deriving this function. Consider

the point in time in which the jth primal constraint is given and assume that it is not satisfied. Our goal is to bound the derivative of the primal cost (denoted by P) as a function of the dual profit (denoted by D). That is, show that

$$\frac{\partial P}{\partial y_j} = \sum_{i \in S(j)} c_i \frac{\partial x_i}{\partial y_j} \leq \alpha \frac{\partial D}{\partial y_j},$$

where α is going to be the competitive factor. Suppose that the derivative of the primal cost satisfies:

$$\sum_{i \in S(j)} c_i \frac{\partial x_i}{\partial y_j} = A \sum_{i \in S(j)} \left(x_i + \frac{1}{d} \right). \tag{4.3}$$

Then, since $\sum_{i \in S(j)} x_i \leq 1$, $\sum_{i \in S(j)} 1/d \leq 1$, and $\partial D / \partial y_j = 1$, we get that

$$A \sum_{i \in S(j)} \left(x_i + \frac{1}{d} \right) \leq 2A \frac{\partial D}{\partial y_j}.$$

Thus, $\alpha = 2A$. Now, satisfying equality (4.3) requires solving the following differential equation for each $i \in S(j)$:

$$\frac{\partial x_i}{\partial y_j} = \frac{A}{c_i} \left(x_i + \frac{1}{d} \right).$$

It is easy to verify that the solution is a family of functions of the following form:

$$x_i = B \cdot \exp\left(\frac{A}{c_i} \sum_{\ell \ | i \in S(j)} y_\ell \right) - \frac{1}{d},$$

where B can take on any value. Next, we have the following two boundary conditions on the solution:

- Initially, $x_i = 0$, and this happens when $1/c_i \sum_{j | i \in S(j)} y_j = 0$.
- If $1/c_i \sum_{j | i \in S(j)} y_j = 1$, (i.e., the dual constraint is tight), then $x_i = 1$. (Then, the primal constraint is also satisfied.)

The first boundary condition gives $B = 1/d$. The second boundary condition gives us $A = \ln(d + 1)$. Putting everything together we get the exact function used in the algorithm.

We next describe a third algorithm. This algorithm is also continuous, yet different from the previous one. The idea is to utilize the complementary slackness conditions in the online algorithm. This idea turns out to be useful in some applications we discuss in later chapters. Again, let $d = \max_j |S(j)| \leq m$ be the maximum size of a constraint. The description of the algorithm is the following:

Algorithm 3:
Upon arrival of a new primal constraint $\sum_{i \in S(j)} x_i \geq 1$ and the corresponding dual variable y_j:

(1) While $\sum_{i \in S(j)} x_i < 1$:
   (a) Increase variable y_j continuously.
   (b) If $x_i = 0$ and $\sum_{j | i \in S(j)} y_j = c_i$ then set $x_i \leftarrow 1/d$.
   (c) For each variable x_i, $1/d \leq x_i < 1$, that appears in the (yet unsatisfied) primal constraint, increase x_i according to the following function:

$$x_i \leftarrow \frac{1}{d} \exp\left(\frac{\sum_{j | i \in S(j)} y_j}{c_i} - 1 \right).$$

First, note that the exponential function equals $1/d$ when $\sum_{j | i \in S(j)} y_j = c_i$ and so the algorithm is well defined. We next prove the following theorem:

Theorem 4.3. The algorithm produces:

- A (fractional) $O(\log d)$-competitive covering solution.
- A (fractional) 2-competitive packing solution and violates each packing constraint by a factor of at most $O(\log d)$.

We remark that similarly to the basic discrete algorithm, it is possible to make the packing solution feasible (and $O(\log d)$-competitive) by scaling down each y_j by a factor of $O(\log d)$.

Proof. Let P and D be the values of the objective function of the primal and dual solutions, respectively. Initially, $P = D = 0$. We prove three simple claims:

(1) The algorithm produces a primal (covering) feasible solution.
(2) Each packing constraint in the dual program is violated by a factor of at most $O(\log d)$.
(3) $P \leq 2D$.

The theorem then follows immediately from the three claims together with weak duality.

Proof of (1): Consider a primal constraint $\sum_{i \in S(j)} x_i \geq 1$. During the jth iteration the algorithm increases the values of the variables x_i until the constraint is satisfied. Subsequent increases of the variables cannot result in infeasibility.

Proof of (2): Consider any dual constraint $\sum_{j|i \in S(j)} y_j \leq c_i$. The corresponding variable x_i cannot exceed 1, since otherwise it would not belong to any unsatisfied constraint. Thus, we get that:

$$x_i = \frac{1}{d} \exp\left(\frac{\sum_{j|i \in S(j)} y_j}{c_i} - 1 \right) \leq 1.$$

Simplifying, we get that

$$\sum_{j|i \in S(j)} y_j \leq c_i(1 + \ln d).$$

Proof of (3): We partition the contribution to the primal objective function into two parts. Let C_1 be the contribution to the primal cost from Step (1b), due to the increase of primal variables from 0 to $1/d$. Let C_2 be the contribution to the primal cost from Step (1c) of the algorithm. It is also beneficial for the reader to observe the similarity between the arguments used for bounding C_1 and those used for the proof of the approximation factor of the primal–dual algorithm in Section 2.2.3.

Bounding C_1: Let $\tilde{x}_i = \min(x_i, 1/d)$. Our goal is to bound $\sum_{i=1}^{n} c_i \tilde{x}_i$. To do this we observe the following. First, the algorithm guarantees

that if $x_i > 0$, and therefore $\tilde{x}_i > 0$, then:

$$\sum_{j|i\in S(j)} y_j \geq c_i \quad \text{(primal complementary slackness)} \tag{4.4}$$

Next, if $y_j > 0$, then:

$$\sum_{i\in S(j)} \tilde{x}_i \leq 1 \quad \text{(dual complementary slackness)} \tag{4.5}$$

Inequality (4.5) follows since $\tilde{x}_i \leq 1/d$. Thus, even if for all i, $\tilde{x}_i = 1/d \leq 1/|S(j)|$, then $\sum_{i\in S(j)} \tilde{x}_i \leq 1$. Note that the inequality holds for any y_j, regardless if $y_j > 0$. By the primal and dual complementary slackness conditions we get that:

$$\sum_{i=1}^{n} c_i \tilde{x}_i \leq \sum_{i=1}^{n} \left(\sum_{j|i\in S(j)} y_j \right) \tilde{x}_i \tag{4.6}$$

$$= \sum_{j=1}^{m} \left(\sum_{i\in S(j)} \tilde{x}_i \right) y_j \tag{4.7}$$

$$\leq \sum_{j=1}^{m} y_j, \tag{4.8}$$

where Inequality (4.6) follows from Inequality (4.4). Equality (4.7) follows by changing the order of summation. Inequality (4.8) follows from Inequality (4.5). Thus, we get that C_1 is at most the dual cost.

Bounding C_2: Whenever the algorithm updates the primal and dual solutions, $\partial D/\partial y_j = 1$. It is easy to verify that the derivative of the primal cost is:

$$\frac{\partial P}{\partial y_j} = \sum_{i\in S(j)} c_i \frac{\partial x_i}{y_j} = \sum_{i\in S(j)} c_i \frac{x_i}{c_i} \leq 1. \tag{4.9}$$

The last inequality follows since the covering constraint is infeasible at the update time. Thus, C_2 is also bounded from above by the dual cost. We conclude: $P = C_1 + C_2 \leq 2D$. □

Remark 4.4. It is possible to even further optimize the performance of Algorithm 3. Initialize each x_i to $\mu = 1/d \ln d$ (instead of $1/d$), and change the continuous update rule to be $x_i \leftarrow \mu \exp\left(\sum_{j|i \in S(j)} y_j/c_i - 1\right)$. This will guarantee that $C_1 + C_2 = (1 + 1/\ln d) \cdot D$, while the dual solution is violated by a factor of $1 + \ln d + \ln \ln d$, yielding that the competitive ratio is $\ln d + \ln \ln d + O(1)$. It is also possible to prove a corresponding lower bound of H_d on the competitive ratio of any online algorithm, meaning that the competitive ratio of the algorithm is tight up to additive terms.

4.3 Lower Bounds

In this section, we show that the competitive ratios obtained in Section 4.2 are optimal up to constants. We prove a lower bound for the online packing problem and another lower bound for the online covering problem.

Lemma 4.5. There is an instance of the online (fractional) packing problem with n constraints, such that for any B-competitive online algorithm, there exists a constraint for which

$$\sum_{j|i \in S(j)} y_j \geq c_i \frac{1}{B}\left(1 + \frac{\log_2 n}{2}\right) = c_i \Omega\left(\frac{\log n}{B}\right).$$

Proof. Consider the following instance with $n = 2^k$ packing constraints. The right-hand side of each packing constraint is 1. In the first iteration, a new variable $y(1,1)$ belonging to all constraints arrives. In the second iteration, two variables $y(2,1)$ and $y(2,2)$ arrive. Variable $y(2,1)$ belongs to the first 2^{k-1} constraints and $y(2,2)$ belongs to the last 2^{k-1} constraints. In the third iteration, four dual variables $y(3,1), y(3,2), y(3,3)$, and $y(3,4)$ arrive belonging each to 2^{k-2} packing constraints. The process ends in the $(k+1)$th iteration in which 2^k variables arrive, each belonging to a single packing constraint. The optimal solution sets the new 2^{i-1} variables in the ith iteration

to 1. Since the algorithm is B-competitive we get the following set of constraints:

For each $1 \leq i \leq k+1$:

$$\sum_{j=1}^{i} \sum_{\ell=1}^{2^{j-1}} y(j,\ell) \geq \frac{2^{i-1}}{B}.$$

Multiplying the $(k+1)$th inequality by 1 and each inequality i, $1 \leq i \leq k$, by 2^{k-i} and summing up, we get that:

$$\sum_{j=1}^{k+1} 2^{k-j+1} \left(\sum_{\ell=1}^{2^{j-1}} y(j,\ell) \right) \geq \frac{1}{B}(k2^{k-1} + 2^k).$$

This follows since for each j, $\left(\sum_{\ell=1}^{2^{j-1}} y(j,\ell)\right)$ is multiplied by $1 + 1 + 2 + 4 + \cdots + 2^{k-j} = 2^{k-j+1}$. However, the left hand side is also the sum over all packing constraints. Thus, by an averaging argument, since there are 2^k constraints, we get that there exists a constraint whose right hand side is at least

$$\frac{1}{2^k \cdot B}\left(k2^{k-1} + 2^k\right) = \frac{1}{B}\left(1 + \frac{k}{2}\right).$$

Since $n = 2^k$ we get the desired bound. □

Lemma 4.6. There is an instance of the online fractional covering problem with n variables such that any online algorithm is $\Omega(\log n)$-competitive on this instance.

Proof. Consider the following instance with $n = 2^k$ variables $x_1, x_2, \ldots, x_n$. The first constraint that arrives is $\sum_{i=1}^{n} x_i \geq 1$. If $\sum_{i=1}^{n/2} x_i \geq \sum_{i=n/2+1}^{n} x_i$, then the next constraint that is given is $\sum_{i=n/2+1}^{n} x_i \geq 1$, otherwise the constraint $\sum_{i=1}^{n/2} x_i \geq 1$ is given. The process of halving and then continuing with the smaller sum goes on until we remain with a single variable. The optimal offline solution satisfying all the constraints sets the last variable to one. However, for any

online algorithm, the variables in each iteration that do not appear in subsequent iterations add up to at least $1/2$. There are $k+1$ iterations, and thus the cost of any online algorithm is at least $1 + k/2$, concluding the proof. □

4.4 Two Warm-Up Problems

In this section, we demonstrate the use of the online primal–dual framework on two simple examples, a covering problem and a packing problem.

4.4.1 The Online Set-Cover Problem

Consider an online version of the offline set-cover problem discussed in Section 2.2. In the online version of the problem, a subset of the elements X arrive one-by-one in an online fashion. The algorithm has to cover each element upon arrival. The restriction is that sets already chosen to the cover by the online algorithm cannot be "unchosen."

This online setting exactly fits the online covering setting, since whenever a new element arrives a new constraint is added to the set-cover linear formulation. Hence, we can use any of the algorithms presented earlier in Section 4.2 to derive a monotonically increasing fractional solution to the set-cover problem.

Getting a randomized integral solution is extremely simple. We can simply adapt the randomized rounding procedure appearing in Section 2.2.2 to the online setting. Note that the algorithm picks *a priori* uniformly in random a threshold $\Theta(s) \in [0,1]$ for each set s. The algorithm then chooses the set s to the cover if $x_s \geq \Theta(s)$. Since x_s is monotonically increasing, the online algorithm simply chooses the set s to the cover once x_s reaches $\Theta(s)$. Note that the number of elements is not known in advance; however, we do choose $\Theta(s)$ by taking the minimum between $O(\log n)$ random variables. Thus, we can increase the number of random variables as the number of elements increases. The threshold $\Theta(s)$ can only decrease by doing that. The analysis is then straightforward proving that the algorithm produces a solution covering all requested elements with high probability, and its expected cost is $O(\log n)$ times the fractional solution. Since the fractional solution is

$O(\log m)$-competitive with respect to the optimal solution, we get that the integral algorithm is $O(\log n \log m)$-competitive.

In Section 5, we show how to obtain a deterministic online algorithm for the set cover problem with the same competitive ratio.

4.4.2 Online Routing

In this section, we give a simple example of an online packing problem. We study the problem of maximizing the throughput of scheduled virtual circuits. In the simplest version of the problem, we are given a graph $G = (V, E)$ with capacities $u(e)$ on the edges. A set of requests $r_i = (s_i, t_i)$ $(1 \leq i \leq n)$ arrives in an online fashion. To serve a request, the algorithm chooses a path between s_i and t_i and allocates a bandwidth of one unit on this path. The decisions of the algorithm are irrevocable, and all requests are permanent, meaning that once accepted they stay forever. If the total capacity routed on edge e is $\ell \cdot u(e)$, we say that the *load* on edge e is ℓ. Ideally, the total bandwidth allocated on any edge should not exceed its capacity (load $\ell \leq 1$). The total profit of the algorithm is the number of requests served and as usual the competitive factor is defined with respect to the maximum number of requests that could have been served offline.

In the fractional version of the problem, the allocation is not restricted to an integral bandwidth equal to either 0 or 1; instead, we can allocate to each request a fractional bandwidth in the range $[0, 1]$. In addition, the bandwidth allocated to a request can be divided between several paths. This problem is an online version of the maximum multi-commodity flow problem. We describe the problem as a packing problem in Figure 4.2. For $r_i = (s_i, t_i)$, let $\mathbb{P}(r_i)$ be the set of

Primal	Dual
Minimize: $\sum_{e \in E} u(e)x(e) + \sum_{r_i} z(r_i)$	Maximize: $\sum_{r_i} \sum_{P \in \mathbb{P}(r_i)} f(r_i, P)$
subject to:	subject to:
$\forall r_i \in \mathbb{R}, P \in \mathbb{P}(r_i)$: $\sum_{e \in P} x(e) + z(r_i) \geq 1$	$\forall r_i \in \mathbb{R}$: $\sum_{P \in \mathbb{P}(r_i)} f(r_i, P) \leq 1$
	$\forall e \in E$: $\sum_{r_i \in \mathbb{R}, P \in \mathbb{P}(r_i) \mid e \in P} f(r_i, P) \leq u(e)$
$\forall r_i, z(r_i) \geq 0$, $\forall e, x(e) \geq 0$	$\forall r_i, P : f(r_i, P) \geq 0$

Fig. 4.2 The splittable routing problem (maximization) and its corresponding primal problem.

simple paths between s_i and t_i. For each $P \in \mathbb{P}(r_i)$, the variable $f(r_i, P)$ corresponds to the amount of flow (service) given to request r_i on the path P. The first set of constraints guarantees that each request gets at most a fractional flow (bandwidth) of 1. The second set of constraints follows from the capacity constraints on the edges. In the primal problem, we assign a variable $z(r_i)$ to each request r_i and a variable $x(e)$ to each edge in the graph.

This online setting exactly fits our online packing setting, as new dual variables arrive whenever a new request arrives. However, in each iteration it may happen that an exponential number of variables arrives. We show in the following that we can still overcome this problem and get an efficient algorithm. We present two algorithms for the problem, each having different properties. Let $d \leq n$ be the length of the longest simple path between any two vertices in the graph. The first algorithm is the following:

Routing algorithm 1:
When a new request $r_i = (s_i, t_i)$ arrives:

(1) if there exists a path $P \in \mathbb{P}(r_i)$ such that $\sum_{e \in P} x(e) < 1$:
  (a) Route the request on P and set $f(r_i, P) \leftarrow 1$.
  (b) Set $z(r_i) \leftarrow 1$.
  (c) For each $e \in P$: $x(e) \leftarrow x(e)(1 + 1/u(e)) + 1/(|P| \cdot u(e))$, where $|P|$ is the length of the path P.

Theorem 4.7. The algorithm is 3-competitive and it violates the capacity of each edge by at most a factor of $O(\log d)$ (i.e., the load on each edge is at most $O(\log d)$).

The proof of Theorem 4.7 is almost identical to the proof of Theorem 4.1, and we leave it as a simple exercise to the reader. The main observation is that the exponential number of dual variables is not obstacle, since the algorithm only needs to check the validity of the condition in line (1). If, for example, $\mathbb{P}(r_i)$ is the set of all simple paths

between s_i and t_i, then this condition can be easily checked by computing a shortest path between s_i and t_i.

The above algorithm may exceed the capacity of the edges. We next show a different algorithm that fully respects the capacities of the edges.

Routing algorithm 2:
Initially: $x(e) \leftarrow 0$.
When a new request $r_i = (s_i, t_i, \mathbb{P}(r_i))$ arrives:

(1) If there exists a path $P(r_i) \in \mathbb{P}(r_i)$ of length < 1 with respect to $x(e)$:
   (a) Route the request on "any" path $P \in \mathbb{P}(r_i)$ with length < 1.
   (b) $z(r_i) \leftarrow 1$.
   (c) For each edge e in $P(r_i)$:

$$x(e) \leftarrow x(e)\exp\left(\frac{\ln(1+n)}{u(e)}\right) + \frac{1}{n}\left[\exp\left(\frac{\ln(1+n)}{u(e)}\right) - 1\right].$$

Theorem 4.8. The algorithm is $O(u(\min)[\exp(\ln(1+n)/u(\min)) - 1])$-competitive and does not violate the capacity constraints. If $u(\min) \geq \log n$ then the algorithm is $O(\log n)$-competitive.

Proof. Note first that the function $(u(e)[\exp(\ln(1+n)/u(e)) - 1])$ is monotonically decreasing with respect to $u(e)$. Thus, when a request r_i is routed, the increase of the primal cost is at most:

$$1 + \sum_{e \in P} u(e)\left(x(e)\left[\exp\left(\frac{\ln(1+n)}{u(e)}\right) - 1\right] + \frac{1}{n}\left[\exp\left(\frac{\ln(1+n)}{u(e)}\right) - 1\right]\right).$$

This expression is at most

$$2\left(u(\min)\left[\exp\left(\frac{\ln(1+n)}{u(\min)}\right) - 1\right]\right) + 1. \tag{4.10}$$

This follows since $z(r_i)$ is set to 1, and edges on the path P satisfy $\sum_{e \in P} x(e) \leq 1$. Each time a request is routed, the dual profit is 1.

Thus, the ratio between the primal and dual solutions is at most Expression (4.10).

Second, observe that the algorithm maintains a feasible primal solution at all times. This follows since $z(r_i)$ is set to 1 for any request for which the distance between s_i and t_i (with respect to the $x(e)$-variables) is strictly less than 1.

Finally, it remains to prove that the algorithm routes at most $u(e)$ requests on each edge e, and so the dual solution it maintains is feasible. To this end, observe that for each edge e, the value $x(e)$ is the sum of a geometric sequence with initial value $1/n[\exp(\ln(1+n)/u(e)) - 1]$ and a multiplier $q = \exp(\ln(1+n)/u(e))$. Thus, after $u(e)$ requests are routed through edge e, the value of $x(e)$ is

$$\begin{aligned} x(e) &= \frac{1}{n}\left(\exp\left(\frac{\ln(1+n)}{u(e)}\right) - 1\right)\frac{\exp\left(\frac{u(e)\ln(1+n)}{u(e)}\right) - 1}{\exp\left(\frac{\ln(1+n)}{u(e)}\right) - 1} \\ &= \frac{1}{n}(1+n-1) = 1. \end{aligned}$$

Since the algorithm never routes requests on edges for which $x(e) \geq 1$, we are done.

Finally, it is not hard to verify that when $u(\min) \geq \log n$, then

$$2\left(u(\min)\left[\exp\left(\frac{\ln(1+n)}{u(\min)}\right) - 1\right]\right) + 1 = O(\log n).$$

□

4.5 Notes

The definitions of the online covering/packing framework along with the basic algorithms in Section 4.2 and the lower bounds in Section 4.3 are based on the work of Buchbinder and Naor [32]. These algorithms draw on ideas from previous algorithms by Alon et al. [3, 4]. The third basic algorithm that incorporates the complementary slackness conditions into the online algorithm is based on the later work of Bansal et al. [14]. The online set-cover problem was considered in [3]. There, they gave a deterministic algorithm for the problem that is discussed later on in Section 5. The second routing algorithm in Section 4.4.2

appeared in [34]. It is basically an alternative description and analysis of a previous algorithm by Awerbuch et al. [11].

There is a long line of work on generating a near-optimal fractional solution for *offline* covering and packing problems, e.g. [52, 53, 54, 55, 75, 79, 84, 93]. All these methods take advantage of the offline nature of the problems. As several of these methods use primal–dual analysis, our approach can be viewed in a sense as an adaptation of these methods to the context of online computation.

5

The Online Set-Cover Problem

In Section 4, we saw how to derive a simple randomized $O(\log m \log n)$-competitive algorithm for the online set-cover problem. An intriguing question is whether we can obtain a deterministic algorithm for the problem with no degradation in the competitive ratio. A common approach for obtaining deterministic algorithms is *derandomization*, which means converting randomized algorithms into deterministic algorithms. For more details on derandomization methods we refer the reader to [6]. We note that one of the fundamental approaches to (offline) derandomization is the so-called method of conditional expectations or pessimistic estimators [6], which performs a deterministic rounding process. Coming up with a function that guides this process (the pessimistic estimator) is the key ingredient to a successful application of the method of conditional expectations.

Fractional solutions can often be converted into randomized algorithms, but it is usually much harder to perform this conversion online. Indeed, this conversion is possible for the online set-cover problem because of the way the fractional solution evolves in time. Furthermore, the randomized algorithm can be converted into a deterministic one by

an *implicit* use of the method of conditional expectations, which leads to the development of a potential function guiding the online algorithm. The competitive factor of the deterministic algorithm obtained remains $O(\log m \log n)$.

5.1 Obtaining a Deterministic Algorithm

If the set system is not known in advance to the algorithm, then it is quite easy to verify that any deterministic algorithm is $\Omega(n)$-competitive. Thus, we assume that the universe of elements X is known to the algorithm along with the family of sets $\mathbb{S}$. It is unknown, however, which subset $X' \subseteq X$ of the elements the algorithm would eventually have to cover (as well as their order of appearance). Let $c(\mathbb{C}_{\mathrm{OPT}})$ denote the cost of the optimal solution. We design an online algorithm that is given a value $\alpha \geq c(\mathbb{C}_{\mathrm{OPT}})$ as input, and produces a feasible solution with cost $O(\alpha \log m \log n)$. However, in case the algorithm is given an infeasible value of α (i.e., $\alpha < c(\mathbb{C}_{\mathrm{OPT}})$), it may fail.

Our algorithm guesses the value of α by doubling. We start by guessing $\alpha = \min_{s \in \mathbb{S}} c_s$, and then run the algorithm with this value. If the algorithm fails we "forget" about all sets chosen so far to $\mathbb{C}$, update the value of α by doubling it, and continue on. We note that the cost of the sets that we have "forgotten" about can increase the cost of our solution by at most a factor of 2, since the value of α doubles in each step. In the final iteration, the value of α can be at most $2c(\mathbb{C}_{\mathrm{OPT}})$, losing altogether a factor of 4 as a result of the doubling procedure. Also, for each choice of α, our algorithm ignores all sets of cost exceeding α, and chooses all sets of cost at most α/m to $\mathbb{C}$.

The algorithm we design uses as a subroutine an online algorithm that generates an $O(\log m)$-competitive fractional solution, e.g., the online fractional algorithm presented in Section 4. This algorithm maintains a monotonically increasing fraction w_s for each set s. Let $w_e = \sum_{s|e \in s} w_s$. Note that the fractional algorithm guarantees that $w_e \geq 1$ for each element e which is requested. Initially, our algorithm starts with the empty cover $\mathbb{C} = \emptyset$. Define C to be the union of all the elements covered by members of $\mathbb{C}$. The following potential function is

used throughout the algorithm:

$$\Phi = \sum_{e \notin C} n^{2w_e} + n \cdot \exp\left(\frac{1}{2\alpha} \sum_{s \in \mathbb{S}} (c_s \chi_{\mathbb{C}}(s) - 3w_s c_s \log n)\right).$$

The function $\chi_{\mathbb{C}}$ above is the characteristic function of $\mathbb{C}$, that is, $\chi_{\mathbb{C}}(s) = 1$ if $s \in \mathbb{C}$, and $\chi_{\mathbb{C}}(s) = 0$ otherwise.

The deterministic online algorithm is as follows:

Run the algorithm presented in Section 4.2 to produce a monotonically increasing online fractional solution. When the weight of set s is increased:

(1) If $s \in \mathbb{C}$ do nothing; Otherwise:
(2) Φ_{start} — value of Φ before increasing the weight of s.
(3) Φ_{end} — value of Φ after increasing the weight of s.
(4) Φ'_{end} — value of Φ after increasing the weight of s and choosing s to the cover.
(5) Choose set s to $\mathbb{C}$ if $\Phi'_{\text{end}} \leq \Phi_{\text{start}}$.
(6) If $\Phi_{\text{start}} > \max\{\Phi'_{\text{end}}, \Phi_{\text{end}}\}$, return "FAIL."

In the following, we analyze the performance of the algorithm in a single iteration in which the condition $\alpha \geq c(\mathbb{C}_{\text{OPT}})$ is satisfied. We prove that the algorithm never fails in this iteration.

Lemma 5.1. Consider a step in which the weight of a set s is augmented by the algorithm. Let Φ_{start} and Φ_{end} be the values of the potential function Φ before and after the step, respectively. Then $\Phi_{\text{end}} \leq \Phi_{\text{start}}$. In particular, when $\alpha \geq c(\mathbb{C}_{\text{OPT}})$ the algorithm never fails.

Proof. Consider first the case in which $s \in \mathbb{C}$. In this case, the first term of the potential function remains unchanged. The second term of the potential function decreases and therefore the lemma holds.

The second case is when $s \notin \mathbb{C}$. The proof for this case is probabilistic. We prove that either including s in $\mathbb{C}$, or not including it, does not increase the potential function. Let w_s and $w_s + \delta_s$ denote the weight

of s before and after the increase, respectively. Add set s to $\mathbb{C}$ with probability $1 - n^{-2\delta_s}$. (This is roughly equivalent to choosing s with probability $\delta_s/2$ and repeating it $4 \log n$ times.)

We first bound the expected value of the first term of the potential function. This is similar to the unweighted case. Consider an element $e \in X$ such that $e \notin C$. If $e \notin s$ then the term that corresponds to this element remains unchanged. Otherwise, let the weight of e before and after the step be w_e and $w_e + \delta_s$, respectively. Before the step, element e contributes to the first term of the potential function the value n^{2w_e}. The probability that we do not choose set s containing element e is $n^{-2\delta_s}$. Therefore, the expected contribution of element e to the potential function after the step is at most $n^{-2\delta_s} n^{2(w_e+\delta_s)} = n^{2w_e}$. By linearity of expectation it follows that the expected value of $\sum_{e \notin \mathbb{C}} n^{2w_e}$ after the step is precisely its value before the step.

It remains to bound the expected value of the second term of the potential function. Let

$$T = n \cdot \exp\left(\frac{1}{2\alpha} \sum_{s \in \mathbb{S}} (c_s \chi_{\mathbb{C}}(s) - 3 w_s c_s \log n)\right)$$

denote the value of the second term of the potential function before the step, and let T' denote the same term after the weight increase and the probabilistic choice of set s to the cover. Recall that $s \notin \mathbb{C}$. Then,

$$\mathbf{Exp}[T'] = T \cdot \exp\left(-\frac{1}{2\alpha} 3\delta_s c_s \log n\right) \cdot \mathbf{Exp}\left[\exp\left(\frac{1}{2\alpha} c_s \chi_{\mathbb{C}'}(s)\right)\right] \tag{5.1}$$

where $\chi_{\mathbb{C}'}(s) = 1$ is the indicator of the event that set s is chosen to the cover, which happens with probability $1 - n^{-2\delta_s}$. We bound the right-hand side of (5.1) as follows:

$$\mathbf{Exp}\left[\exp\left(\frac{1}{2\alpha} c_s \chi_{\mathbb{C}'}(s)\right)\right] = n^{-2\delta_s} + (1 - n^{-2\delta_s}) \cdot \exp\left(\frac{c_s}{2\alpha}\right) \tag{5.2}$$

$$\leq 1 - 2\delta_s \log n + 2\delta_s \log n \exp\left(\frac{c_s}{2\alpha}\right) \tag{5.3}$$

$$= 1 + 2\delta_s \log n \left(\exp\left(\frac{c_s}{2\alpha}\right) - 1\right) \tag{5.4}$$

$$\leq 1 + 2\delta_s \log n \frac{3c_s}{4\alpha} \tag{5.5}$$

$$\leq \exp\left(\frac{3\delta_s c_s \log n}{2\alpha}\right). \tag{5.6}$$

Here, (5.3) follows since for all $x \geq 0$ and $z \geq 1$, $e^{-x} + (1 - e^{-x}) \cdot z \leq 1 - x + x \cdot z$, (5.5) follows[1] since $e^y - 1 \leq 3y/2$ for all $0 \leq y \leq 1/2$, and (5.6) follows since $1 + x \leq e^x$ for all $x \geq 0$. Plugging in (5.1), we conclude that the expected value of the second term after the increase step and the probabilistic choices is at most

$$\mathbf{Exp}[T'] = T \cdot \exp\left(-\frac{1}{2\alpha} 3\delta_s c_s \log n\right) \cdot \exp\left(\frac{1}{2\alpha} 3\delta_s c_s \log n\right) \leq T.$$

By linearity of expectation it now follows that $\mathbf{Exp}[\Phi_{\text{end}}] \leq \Phi_{\text{start}}$. Therefore, either the event of choosing s to the cover, or the event of not choosing s to the cover, does not increase the potential function. We conclude that after each step $\Phi_{\text{end}} \leq \Phi_{\text{start}}$. □

Theorem 5.2. Throughout the algorithm, the following holds:

(i) Every $e \in X$ of weight $w_e \geq 1$ is covered, that is, $e \in C$.
(ii) $\sum_{s \in \mathbb{C}} c_s = \alpha \cdot O(\log m \log n)$.

Proof. Initially, the value of the potential function Φ is at most $n \cdot n^0 + n < n^2$, and hence it remains smaller than n^2 throughout the whole algorithm. Therefore, if for element e, $w_e \geq 1$, at some point of time, then $e \in C$, since otherwise the contribution of the term n^{2w_e} itself would be at least n^2. This proves part (i). To prove part (ii), note that by the same argument, throughout the algorithm

$$n \cdot \exp\left(\frac{1}{2\alpha} \sum_{s \in \mathbb{S}} c_s \chi_{\mathbb{C}}(s) - 3w_s c_s \log n\right) < n^2.$$

[1] Note that here we use the fact that $\alpha \geq c(\mathbb{C}_{\text{OPT}})$, since we ignored all sets with cost greater than α.

Therefore,

$$\sum_{s \in \mathbb{S}} c_s \chi_{\mathbb{C}}(s) \leq \sum_{s \in \mathbb{S}} 3 w_s c_s \log n + 2\alpha \log n,$$

and the desired result follows from the fact that the fractional solution is $O(\log m)$-competitive. □

5.2 Notes

The results in this section are based on the work of Alon et al. [3]. We note that the algorithms of [3] were not originally stated as primal–dual algorithms, yet interpreting them as primal–dual algorithms was the starting point of extending the primal–dual method to the realm of online computation. Alon et al. [3] also proved that any deterministic algorithm for the online set-cover problem is $\Omega(\log n \log m/(\log \log m + \log \log n))$-competititive for many values of m and n. In the unweighted version of the set-cover problem all sets are of unit cost and so the goal is to minimize the number of sets needed for covering the elements. Buchbinder and Naor [32] used the improved offline rounding technique of [89] to obtain an $O(\log d \log(n/\mathrm{OPT}))$-competitive algorithm, where d is the maximum frequency of an element (i.e., the maximum number of sets an element belongs to), n is the number of elements and OPT is the optimal size of the set cover.

Derandomization has proved to be useful for other online problems as well. Buchbinder and Naor [32], using derandomization methods, obtained an alternative routing algorithm that achieves the same competitiveness as the second routing algorithm in Section 4.4.2. The algorithm is based on the basic algorithms of Section 4.2 along with an online derandomization of the rounding method in [85, 86]. Another example of a derandomization of an online algorithm is by Buchbinder et al. [30] for the ad-auctions problem (see also Section 10). However, we note that we do not yet fully understand when derandomization methods can be applied in online settings. For example, for the online group Steiner problem on trees discussed in Section 11, there is currently only a randomized online algorithm, even though the offline randomized algorithm can be derandomized.

Another online variation of the set cover problem was considered in [10]. There, we are also given m sets and n elements that arrive one at a time. However, the goal of the online algorithm is to pick k sets so as to maximize the number of elements that are covered. The algorithm only gets credit for elements that are contained in a set that it selected *before* or *during* the step in which the element arrived. The authors of [10] showed a *randomized* $\Theta(\log m \log(n/k))$ competitive algorithm for the problem, where the bound is optimal for many values of n, m, and k. A different extension of the online set cover problem is studied in [5]. They considered an admission control problem where the goal is to minimize the number of rejections. The problem is solved by reducing it to an instance of online set cover with repetitions, i.e., each element may need to be covered several times.

6

The Metrical Task System Problem on a Weighted Star

In this section, we study the metrical task system (MTS) problem on a metric $\mathbb{M}$ defined by a weighted star. The MTS is one of the earliest problems studied in the context of online computation. The problem captures many online scenarios. In the MTS problem, we are given a finite metric space $\mathbb{M} = (V, d)$, where $|V| = N$. We view the points of $\mathbb{M}$ as states to which the algorithm belongs. The distance between the points of the metric measures the cost of transition between the possible states. We use a "server" notation and say that there is a server moving between the states and serving the requests. Each task (request) r in a metrical task system is associated with a vector $(r(1), r(2), \ldots, r(N))$, where $r(i)$ denotes the cost of serving r in state $i \in V$. In order to serve request r in state i the server has to be in this state. Upon arrival of a new request, the state of the system can first be changed to a new state (paying the transition cost), and only then the request is served (paying for serving the request in the new state). The transition cost between the states is assumed to be a metric. The objective is to minimize the total cost which is the sum of the transition costs and the service costs.

There is also an equivalent continuous time MTS model. In this model, the algorithm is allowed to change states at any time t, which

is a real number, and not only at integral times. The service cost is generalized in a straight forward way to be an integral instead of a sum. It is well known [28, Section 9.1.1] that any continuous time algorithm can be transformed to a discrete time algorithm without increasing the total cost. On the other hand, since the continuous time model is a relaxation of the discrete model it is clear that the optimal cost cannot increase.

We show in this section how to design an optimal online algorithm for the case where the metric space is a weighed star. The idea is to define an alternative MTS model and show that it is "equivalent" up to constant factors to the original model on a weighted star metric. We then show that the basic algorithms presented in Section 4.2 are applicable to the new model. Finally, we show that a randomized algorithm can be obtained by a simple rounding technique.

The leaves of the star are denoted by $\{1, 2, \ldots, N\}$ and the distance from each leave i to the central state is denoted by $d'(i)$. We present an $O(\log N)$-competitive online algorithm. We are going to charge the algorithm by $2d'(i)$ whenever the server moves from state i to another state, say j. Thus, we are not going to charge the algorithm for the cost of moving into state j. This assumption can only add an additive term to the total cost which is independent of the request sequence (we do not charge for the last state change). From now on we only use $d(i) = 2d'(i)$ to denote the cost of moving from state i to any other state.

6.1 A Modified Model

We start with the (equivalent) continuous time MTS model. In this model, the algorithm is allowed to change states at any time t which is a real number and not only at integral times. We first define a new MTS model and show that on a star metric the cost of an optimal solution can only change by a constant factor. The high-level idea of the new model is to cancel the transition cost incurred due to state change and pay only for serving the requests. To this end, to balance between transition cost and service cost, we restrict the algorithm and allow it to change its state only if certain conditions are fulfilled. For each state, we partition the time interval into phases. We permit the solutions to

leave a state i only at the end of a phase (of state i). The first phase of each state starts at time $t = 0$. Phase p of state i starts at time $t_{p-1}(i)$ and ends at the earliest time $t_p(i)$ for which the accumulated cost of service at state i in the interval $[t_{p-1}(i), t_p(i)]$ is exactly $d(i)$.

We are now ready to describe the new MTS model in its full generality. An online algorithm is allowed to leave state i only at the end of a phase (of state i). The algorithm does not pay any transition cost when moving from one state to another. If the algorithm is in state i during phase p then it pays a cost $d(i)$. The algorithm pays the full cost of the phase even if it was in state i only during part of the phase p. This can happen if the algorithm moves to state i from i' in the middle of the pth phase of i (and at the end of a phase of state i'). Given a set of requests $\bar{\sigma}$, let $\mathrm{OPT}_n(\bar{\sigma})$ be the minimum offline cost of serving the set of requests in the new MTS model. Let $\mathrm{OPT}_o(\bar{\sigma})$ be the minimum offline cost of serving the set of requests in the standard MTS model. We prove the following two lemmas:

Lemma 6.1. Let $\bar{\sigma}$ be a set of requests. Any solution S to $\bar{\sigma}$ in the standard MTS model with cost C can be transformed into a legal solution S' in the new MTS model with cost at most $2C$. In particular, $\mathrm{OPT}_n(\bar{\sigma}) \leq 2\mathrm{OPT}_o(\bar{\sigma})$.

Proof. Let $t_1, t_2, \ldots, t_k$ be the times at which solution S changes its state. Let s_i be the state of the algorithm from time t_{i-1} until time t_i. During this time, the algorithm pays for the cost for serving the requests in state s_i and then pays for moving out of s_i at t_i. We define a solution S' in the new model that imitates S but changes state only at the end of a phase. Initially, S' starts out from the same state s_1 as solution S. At any time t, if the algorithm is in state j in solution S', it waits until the some $t' \geq t$, when the phase in state j ends and then moves to the state in which solution S is at time t'. Note that if S' is already in the same state as S at time t', then S' does not change its state. Clearly, S' is a feasible solution in the new MTS model by design as it changes its state only at the end of a phase. Also, it is easily seen that if solution S' is in state i at some time during $[t_{i-1}, t_i]$, then it stays in i at least until time t_i.

We bound the cost of S'. Let W_i be the total cost of serving the requests in state s_i during the time interval $[t_{i-1}, t_i]$. The cost of the solution S during $[t_{i-1}, t_i]$ is, therefore, $W_i + d(s_i)$. By construction, if S' moves to state s_i during $[t_{i-1}, t_i]$, then it leaves s_i no earlier than time t_i. The extra cost of solution S' with respect to S comes from two sources. First, solution S' may leave state s_i after time t_i. Second, solution S' may move to s_i in the middle of a phase (of state s_i), but it still pays the full cost of the phase. However, each of these can only increase the cost of solution S' by at most $d(s_i)$. Recall, that in the new MTS model the solution does not pay for changing the state, and thus its cost is at most $W_i + 2d(s_i)$, which is at most twice the cost incurred by solution S. □

Lemma 6.2. Let $\bar{\sigma}$ be a set of requests. Any solution S to $\bar{\sigma}$ in the new MTS model with cost C is a legal solution S' in the standard MTS model with cost at most $2C$.

Proof. We run the solution S in the standard MTS model and upper bound its cost in this model. First, the cost of serving the requests in the standard MTS model is no more than the cost of serving the requests in the new model. Suppose that solution S visits state i during phase p. In this case, it pays at least $d(i)$ in the new model, while the cost in the standard model is at most $d(i)$ (the cost could lower if S did not stay in state i during the entire phase).

Second, we claim that the transition cost of solution S in the standard model is no more than the service cost of S in the new model. This follows as S leaves any state i at most once during its phase (at the end of the phase) and hence the cost $d(i)$ of leaving the state can be charged to the service cost of the corresponding phase that just ended (which is also exactly $d(i)$). Thus, the cost of the solution S in the standard MTS model is at most twice its cost in the new MTS model. □

From Lemmas 6.1 and 6.2, a c-competitive algorithm in the new MTS model implies a $4c$-competitive algorithm in the standard MTS model, and hence it suffices to consider the new MTS model.

6.2 The Algorithm

We next describe a simple linear programming formulation for the offline problem in the new MTS model. Our online algorithm will generate a fractional solution to this linear program. We later show how to transform this fractional solution to a randomized integral solution. Let $x(i,p)$ be an indicator to the event that the solution is in state i during the pth phase. We relax the solution and allow the algorithm to be at time t in several states as long as the sum of the fractions of the states is at least 1. (The latter constraint is valid since our objective function is minimization.) Let n_i be the number of phases of state i. The linear program is as follows:

$$(P) \qquad \min \sum_{1=1}^{N} \sum_{p=1}^{n_i} d(i)x(i,p).$$

For any time t:

$$\sum_{i=1}^{N} \sum_{p \mid t \in [t_{p-1}(i), t_p(i)]} x(i,p) \geq 1. \tag{6.1}$$

It may seem that the linear program contains an unbounded number of constraints. However, it is easy to see that we need only to consider times t which are the end of a phase for some state. It can also be easily verified that given an instance of the MTS problem, any feasible solution in the new MTS model defines a feasible solution to (P) with the same cost. We also observe that a feasible solution to (P) defines a (fractional) solution which is feasible in the new MTS model with the same cost. We should be a bit more careful in the online case, where the constraints of (P) are revealed one-by-one. Upon arrival of a constraint, the algorithm finds a feasible assignment to the (primal) variables that satisfies the constraint. Consider variable $x(i,p)$. In the offline case, we can assume without loss of generality that the value of $x(i,p)$ is determined at the beginning of phase p of state i. However, this is not necessarily true in the online case; thus, we restrict our attention to solutions that assign values to $x(i,p)$ forming a monotonically non-decreasing sequence.

The above formulation of the problem is now a covering linear program and thus it fits the online primal–dual framework. We now can apply the algorithms from Section 4.2 to derive a monotonically increasing fractional solution. The algorithm produces an $O(\log N)$-competitive solution since the number of variables in each covering constraint is exactly N.

Rounding the fractional solution: Rounding the fractional solution for this problem is simple. The algorithm maintains the invariant that it is in state i at time t (in phase p) with probability equal to $x(i,p)$. Suppose that at the end of phase p of state i, $x(i,p) = a$. The distribution mass a is then distributed among the states of the system (including i) with respect to the fractional solution. Let a_j be the increase of the fraction associated with state j at that point of time. As $\sum_j a_j = a$, if the algorithm was in state i at the end on phase p, it moves to state j with probability a_j/a. It is easy to verify that the expected cost of the algorithm is exactly the cost of the fractional solution.

6.3 Notes

The results in this chapter are based on the work of Bansal et al. [14]. The MTS problem has been studied extensively. The MTS model was originally formulated by Borodin et al. [29] who gave tight upper and lower bound of $2N - 1$ for any deterministic online algorithm for the problem. They also designed a $2H_N$-competitive randomized algorithm for the uniform metric, and showed a lower bound of H_N for this metric. In fact, our proposed algorithm for the weighed star can be seen in retrospect as a direct generalization of their approach. For the MTS problem on a weighted star, Blum et al. [25] gave a randomized $O(\log^2 N)$-competitive algorithm. For general metrics, Bartal et al. [19] designed a randomized $O(\log^5 N)$-competitive algorithm which is based on an algorithm for hierarchically well-separated trees (HSTs). Fiat and Mendel [49] improved this bound and designed an $O(\log^2 N \log\log N)$-competitive algorithm for general metrics. Fiat and Mendel [49] have also designed an $O(\log N)$-competitive algorithm for the MTS problem

on a weighted star metric. However, their algorithm is based on the fact that a weighted star metric can be approximated by an HST having certain nice properties. Thus, it is not a "direct" algorithm for the problem, in contrast to the algorithm we have presented in this section.

7

Generalized Caching

Caching is one of the earliest and most effective techniques of accelerating the performance of computing systems. Thus, vast amounts of work have been invested in the improvement and refinement of caching techniques and algorithms. In the classic two-level caching problem, we are given a collection of n pages and a fast memory (cache) which can hold up to k of these pages. At each time step one of the pages is requested. If the requested page is already in the cache then no cost is incurred, otherwise the page must be brought into the cache, possibly evicting some other page, and a cost of one unit is incurred. This basic model can be extended in two orthogonal directions. First, the cost of bringing a page into the cache may not necessarily be uniform for all pages. This version of the problem is called *weighted caching* and it models scenarios in which the cost of fetching a page is not the same for all pages due to different locations of the pages (e.g., main memory, disk, web). Second, the sizes of the pages need not be uniform. This is motivated by web caching where pages have varying sizes. Web caching is an extremely useful technique for enhancing the performance of World Wide Web applications. Since fetching a web page or any other information from the internet is usually costly, it is common practice to

keep some of the pages closer to the client. This is done, for example, by the web browser itself by keeping some of the pages locally, and also by internet providers that maintain proxy servers for exactly the same purpose.

We study here several models in which pages have non-uniform sizes. The most general setting is called the *general model* in which pages have both non-uniform sizes and non-uniform fetching costs. Two commonly studied special cases are the so-called *bit model* and *fault model*. In the bit model, the cost of fetching a page is proportional to its size, thus minimizing the fetching cost corresponds to minimizing the total traffic in the network. In the fault model, the fetching cost is uniform for all pages, thus corresponding to the number of times a user has to wait for a page to be retrieved.

Our solutions are based on a two-step approach. In the first step, we obtain an $O(\log k)$-competitive *fractional* algorithm which is based on an online primal–dual approach. In the second step, we obtain a randomized algorithm by rounding online the fractional solution to an actual distribution on integral cache solutions.

In Sections 7.1 and 7.2 we study the weighted caching problem. Later on in Sections 7.3 and 7.4 we extend the ideas used for the weighted caching problem to the more general setting in which pages have both non-uniform sizes and non-uniform fetching costs.

7.1 The Fractional Weighted Caching Problem

In this section, we study the weighted caching problem. In the weighted caching problem, each page p is associated with a positive fetching cost $c_p \geq 1$, denoting the cost of fetching the page into the cache. A request sequence is a sequence of pages, denoted by $p_1, p_2, \ldots$, where page p_t is requested at time t. The tth request is served if page p_t belongs to the cache at time t, for each $t \geq 1$. The objective is to minimize the total cost of fetching pages into the cache.

7.1.1 Linear Programming Relaxation

Consider the following integer program for the (offline) weighted caching problem. Instead of charging for fetching pages into the cache

we charge for evicting them, changing the cost by at most an additive term, which is independent of the request sequence, in any cache replacement policy. (Fetching the last k pages is for "free", since these pages are not evicted from the cache.) Let $x(p,j)$ be an indicator variable for the event that page p is evicted from the cache between the jth time it is requested and the $(j+1)$th time it is requested. If $x(p,j)=1$, we can assume without loss of generality that page p is evicted in the first time slot following the jth time it is requested. (As we later discuss, this assumption is not necessarily true in the online case.) For each page p, denote by $t(p,j)$ the time it is requested for the jth time, and denote by $r(p,t)$ the number of times page p is requested until time t (including t). For any time t, let $B(t)=\{p \mid r(p,t)\geq 1\}$ denote the set of pages that have been requested until time t (including t). Let p_t be the page that is requested at time t. We need to satisfy the constraint that at any time t, the total space used by pages in $B(t)$ is at most k. Now, one unit of space in the cache is already taken up by p_t, implying that a space of at most $k-1$ units can be used by pages in $B(t)\setminus\{p_t\}$. Equivalently, pages in $B(t)\setminus\{p_t\}$ with cumulative size of at least $|B(t)|-1-(k-1)=|B(t)|-k$ must be absent from the cache at time t. This gives the following exact formulation of the problem:

$$\min \sum_{p=1}^{n} \sum_{j=1}^{r(p,T)} c_p \cdot x(p,j).$$

For any time t:

$$\sum_{p\in B(t)\setminus\{p_t\}} x(p,r(p,t)) \geq |B(t)|-k.$$

For any p,t:

$$x(p,t)\in\{0,1\}.$$

In a fractional solution, we relax the constraint for $x(p,t)$ allowing it to assume any value between 0 and 1, i.e., $0\leq x(p,t)\leq 1$.

In the dual program, there is a variable $y(t)$ for each time t and a variable $z(p,j)$ for each page p and the jth time it is requested. The dual

program is the following:

$$\max \sum_{t} (|B(t)| - k)\, y(t) \sum_{p=1}^{n} \sum_{j=1}^{r(p,t)} z(p,j).$$

For each page p and the jth time it is requested:

$$\left(\sum_{t=t(p,j)+1}^{t(p,j+1)-1} y(t) \right) - z(p,j) \leq c_p. \tag{7.1}$$

For any p, j:

$$z(p,j) \geq 0.$$

For all t:

$$y(t) \geq 0.$$

7.1.2 A Fractional Primal–Dual Algorithm

Our online caching algorithm produces fractional primal and dual solutions to the linear formulation. In the online case, the constraints of LP (corresponding to the requests to pages) are revealed one-by-one. Upon arrival of a constraint, the algorithm finds a feasible assignment to the (primal) variables that satisfies the constraint. Consider variable $x(p,j)$. In the offline case, we can assume without loss of generality that the value of $x(p,j)$ is determined at time $t(p,j)+1$. However, this is not necessarily true in the online case; thus, we stipulate that the values assigned to $x(p,j)$ in the time interval $[t(p,j)+1, t(p,j+1)-1]$ by the online algorithm form a monotonically non-decreasing sequence.

We start with a high-level description of the algorithm. Upon arrival of a new constraint at time t, if it is already satisfied, then the algorithm does nothing. Otherwise, the algorithm needs to satisfy the current constraint by increasing some of the primal variables in the constraint. Satisfying the constraint guarantees that there is a enough space in the cache for the new page. To this end, the algorithm starts increasing (continuously) the new dual variable $y(t)$. This, in turn, tightens some of the dual constraints corresponding to primal variables $x(p,j)$ whose current value is 0. Whenever such an event happens, the value of $x(p,j)$

is increased from its initial setting of 0 to $1/k$. Thus, during the time preceding the increase of $x(p,j)$ from 0 to $1/k$, page i cannot be evicted from the cache. This part is somewhat similar to what happens in the randomized marking algorithm of [48]. Meanwhile, variables $x(p,j)$ which are already set to $1/k$ are increased (continuously) according to an exponential function of the new dual variable $y(t)$. Note that this exponential function is equal to $1/k$ when the constraint is tight. Thus, the algorithm is well defined. When variable $x(p,j)$ reaches 1, the algorithm starts increasing the dual variable $z(p,j)$ at the same rate as $y(t)$. As a result, from this time on, the value of $x(p,j)$ remains 1. The algorithm is presented in a continuous fashion, but it can easily be implemented in a discrete fashion. The algorithm is the following:

Fractional Caching algorithm: At time t, when page p_t is requested:

- Set the new variable: $x(p_t, r(p_t,t)) \leftarrow 0$. (It can only be increased in times $r(p_t,t) < t' < r(p_t,t)+1$.)
- If the primal constraint corresponding to time t is satisfied, then do nothing.
- Otherwise: increase primal and dual variables, until the primal constraint corresponding to time t is satisfied, as follows:

  (1) Increase variable $y(t)$ continuously;
  for each variable $x(p,j)$ that appears in the (yet unsatisfied) primal constraint that corresponds to time t:

  (2) If $x(p,j) = 1$, then increase $z(p,j)$ at the same rate as $y(t)$.

  (3) If $x(p,j) = 0$ and

$$\left(\sum_{t=t(p,j)+1}^{t(p,j+1)-1} y(t) \right) - z(p,j) = c_p,$$

  then set $x(p,j) \leftarrow 1/k$.

(*Continued*)

(*Continued*)

(4) If $1/k \leq x(p,j) < 1$, increase $x(p,j)$ according to the following function:

$$\frac{1}{k} \exp\left(\frac{1}{c_p}\left[\left(\sum_{t=t(p,j)+1}^{t(p,j+1)-1} y(t)\right) - z(p,j) - c_p\right]\right).$$

Note that the exponential function for $x(p,j)$ contains variables $y(t)$ that correspond to future times. However, these variables are all initialized to 0, so they do not contribute to the value of the function.

We also remark that setting variable $x(p,j)$ to 1 instead of $1/k$ in line (3), and removing line (4) from the algorithm, results in a deterministic k-competitive algorithm for the weighted caching problem. This turns out to be exactly Young's dual-greedy algorithm [43, 92].

The analysis of the primal cost is partitioned into two parts. The first one corresponds to the contribution of the increase of the variables $x(p,j)$ from 0 to $1/k$, and the second part corresponds to the increase of the variables $x(p,j)$ from $1/k$ to (at most) 1, according to the exponential function. Each part is upper bounded separately by the dual solution, yielding the desired result. We now prove the following theorem.

Theorem 7.1. The fractional caching algorithm is $O(\log k)$-competitive. Specifically, the algorithm is $2(1 + \ln k)$-competitive.

Remark 7.2. It is possible to slightly improve the competitive ratio of the algorithm from $2(1 + \ln k)$ to approximately $\ln k$. To do so, simply replace $1/k$ in lines (3) and (4) in the algorithm by $1/k \ln k$. It is not hard to verify in the proof of Theorem 7.1 that this will result in an algorithm with competitive ratio $\ln k$ plus lower-order terms (e.g. $\ln \ln k$). In the rounding phase, however, several more constants are lost so overall this optimization is not worthwhile.

Proof. [Proof of Theorem 7.1]: First, we note that the primal solution generated by the algorithm is feasible. This follows since, in each

iteration, the variables $x(p,j)$ are increased until the new primal constraint is satisfied. Also, each variable $x(p,j)$ is never increased to be greater than 1. Since, whenever $x(p,j)$ increases in some round and reaches 1, the algorithm starts increasing $z(p,j)$ at the same rate as $y(t)$. Therefore, the value of $x(p,j)$ is not going to change anymore, as the exponent of the exponential function will not change any more. This also implies that the dual solution that we generate is almost feasible, since the dual constraint corresponding to page p and the jth time it is requested satisfies:

$$x(p,j) = \frac{1}{k} \exp\left(\frac{1}{c_p}\left[\left(\sum_{t=t(p,j)+1}^{t(p,j+1)-1} y(t)\right) - z(p,j) - c_p\right]\right) \leq 1.$$

Simplifying, we get that:

$$\left(\sum_{t=t(p,j)+1}^{t(p,j+1)-1} y(t)\right) - z(p,j) \leq c_p(1 + \ln k). \tag{7.2}$$

Thus, the dual solution can be made feasible by scaling it down by a factor of $(1 + \ln k)$. We now prove that the primal cost is at most twice the dual profit, which means that the primal solution produced by the algorithm is $2(1 + \ln k)$-competitive.

We partition the primal cost into two parts. Let C_1 be the contribution to the primal cost from line (3) of the algorithm, due to the increase of variables $x(p,j)$ from 0 to $1/k$. Let C_2 be the contribution to the primal cost from step (4) of the algorithm, due to the incremental increase of the variables $x(p,j)$ from $1/k$ up to at most 1 according to the exponential function.

Bounding C_1: Let $\tilde{x}(i,j) = \min(x(p,j), 1/k)$. We bound the term $\sum_{i=1}^{n} \sum_{j=1}^{r(p,t)} c_p \tilde{x}(i,j)$. To this end, we need several observations. First, if $x(p,j) > 0$, and equivalently if $\tilde{x}(i,j) > 0$, then:

$$\left(\sum_{t=t(p,j)+1}^{t(p,j+1)-1} y(t)\right) - z(p,j) \geq c_p. \tag{7.3}$$

We shall refer to (7.3) as the *primal complementary slackness* condition.

Next, at time t let $B'(t)$ be the set of pages $p \in B(t)$ such that $x(p, r(p,t)) = 1$. In the dual solution, if $y(t)$ is being increased at time t then:

$$\sum_{p \in B(t) \setminus (B'(t) \cup \{p_t\})} \tilde{x}(p, r(p,t)) \leq |B(t)| - k - |B'(t)|. \tag{7.4}$$

We shall refer to (7.4) as the *dual complementary slackness*. Inequality (7.4) follows since there are $|B(t)| - 1 - |B'(t)|$ variables in the constraint corresponding to t. By definition, for each p, $\tilde{x}(p, r(p,t)) \leq 1/k$. Thus, even if for all p, $\tilde{x}(p, r(p,t)) = 1/k$, then the left-hand side of (7.4) adds up to $|B(t)| - 1 - |B'(t)|/k \leq |B(t)| - k - |B'(t)|$. The latter inequality holds since $|B(t)| - |B'(t)| \geq k + 1$, since otherwise the constraint at time t is already satisfied and the algorithm stops increasing the variable $y(t)$. Also, it follows from the algorithm that if $z(p,j) > 0$, then:

$$x(p,j) \geq 1. \tag{7.5}$$

We shall refer to (7.5) as the second dual complementary slackness condition. The primal and dual complementary slackness conditions imply the following:

$$\begin{aligned}
&\sum_{p=1}^{n} \sum_{j=1}^{r(p,t)} c_p \tilde{x}(p,j) \\
&\quad \leq \sum_{p=1}^{n} \sum_{j=1}^{r(p,t)} \left(\left(\sum_{t=t(p,j)+1}^{t(p,j+1)-1} y(t) \right) - z(p,j) \right) \tilde{x}(p,j) && (7.6) \\
&\quad = \sum_{t} \left(\sum_{i \in B(t) \setminus \{p_t\}} \tilde{x}(p, r(p,t)) \right) y(t) - \sum_{p=1}^{n} \sum_{j=1}^{r(p,t)} \tilde{x}(p,j) z(p,j) && (7.7) \\
&\quad \leq \sum_{t} (|B(t)| - k)\, y(t) - \sum_{p=1}^{n} \sum_{j=1}^{r(p,t)} z(p,j). && (7.8)
\end{aligned}$$

Inequality (7.6) follows from inequality (7.3), equality (7.7) follows by changing the order of summation. To see why inequality (7.8) holds consider some time t. Consider the derivative of the left-hand side at

time t. By the algorithm (inequality (7.5)), we increase $z(p,j)$ at the same rate as $y(t)$ only when $x(p,r(p,t)) = 1$ and so $p \in B'(t)$. Thus, the derivative of the left-hand side is $\sum_{p \in B(t) \setminus (B'(t) \cup \{p_t\})} \tilde{x}(p,r(p,t))$. By inequality (7.4), this sum is at most $|B(t)| - k - |B'(t)|$ which is exactly the derivative of the right-hand side of the inequality. Thus, C_1 is at most the profit of a *feasible* dual solution multiplied by $(1 + \ln k)$.

Bounding C_2: We bound the derivative of the increase of variables $x(p,j)$ in line (4) by the derivative of the dual profit accrued in the same round. In each round, only variables $x(p,j)$ that belong to the new primal constraint (and correspond to the new dual variable $y(t)$) are being increased. However, variables $x(p,j)$ that belong to the new primal constraint but have already reached the value of 1 are not increased anymore, and so do not contribute to the primal cost. In the dual program, the new variable $y(t)$ is raised with rate 1, and also all the variables $z(p,j)$ that correspond to variables $x(p,j)$ (in the new primal constraint) that are already equal to 1. It is useful for the purpose of the analysis to think of the latter process as increasing a time variable τ, and then raising the variable $y(t)$ and the appropriate variables $z(p,j)$ with rate 1 with respect to the time variable τ. Using this notation we get that:

$$\begin{aligned}
\frac{\mathrm{d}C_2}{\mathrm{d}\tau} &= \sum_{p \in B(t) \setminus \{p_t\}, 1/k \le x(p,j) < 1} c_p \cdot \frac{\mathrm{d}x(p,r(p,t))}{\mathrm{d}y(t)} \cdot \frac{\mathrm{d}y(t)}{\mathrm{d}\tau} \\
&= \sum_{p \in B(t) \setminus \{p_t\}, 1/k \le x(p,j) < 1} x(p,r(p,t)) \qquad (7.9) \\
&\le (|B(t)| - k) - \sum_{p \in B(t) \setminus \{p_t\}, x(p,j) = 1} 1 \qquad (7.10) \\
&= (|B(t)| - k) \frac{\mathrm{d}y(t)}{\mathrm{d}\tau} - \sum_{p \in B(t) \setminus \{p_t\}, x(p,j) = 1} \frac{\mathrm{d}z(p,j)}{\mathrm{d}\tau}.
\end{aligned}$$

Equality (7.9) follows since $\mathrm{d}y(t)/\mathrm{d}\tau = 1$ and also $\mathrm{d}x(p,j)/\mathrm{d}y(t) = 1/c_p \cdot x(p,j)$ for each $x(p,j)$, $1/k \le x(p,j) < 1$. Inequality (7.10) holds

since the new primal constraint is not yet satisfied and thus:

$$\sum_{p \in B(t) \setminus \{p_t\}, 1/k \leq x(p,j) < 1} x(p, r(p,t)) + \sum_{p \in B(t) \setminus \{p_t\}, x(p,j)=1} x(p, r(p,t)) \; < \; |B(t)| - k.$$

We also remark that by the properties of the algorithm, any variable $x(p,j)$ which is strictly less than $1/k$ is actually equal to 0. Finally, the last term exactly equals the derivative of the dual profit with respect to τ. Therefore, the change in the dual profit is greater than or equal to the change in C_2. Thus, C_2 is at most the profit of a *feasible* dual solution multiplied by $(1 + \ln k)$.

Completing the analysis: It follows that $C_1 + C_2$ is at most twice the profit of a *feasible* dual solution multiplied by $(1 + \ln k)$. Note that the profit of any dual feasible solution is always a lower bound on the optimal solution. Therefore, we conclude by weak duality that the algorithm is $2(1 + \ln k)$-competitive. □

7.1.3 A Fractional Algorithm for the Weighted (h,k)-Caching Problem

A common approach to proving better performance of an online caching algorithm is the (h,k)-caching problem where an online algorithm with cache size k is compared to an offline algorithm with cache size h. In this section, we show a simple modification of the algorithm for this setting.

The modified online algorithm generates a primal solution to a linear program in which the cache size is k. However, the algorithm generates a dual solution which corresponds to a linear program in which the cache size is $h \leq k$. We will perform a primal–dual analysis and show that the primal cost is no more than $O(\log(k/(k-h+1)))$ times the dual cost:

$$\sum_t (|B(t)| - h)\, y(t) \; - \; \sum_{p=1}^{n} \sum_{j=1}^{r(p,t)} z(p,j).$$

Since the dual cost is a lower bound on the offline cost with cache size h, the desired result follows. For convenience, let η denote $(k-h+1)/k$.

To avoid trivialities, we assume that $k \geq h$ and $h > 1$, implying that $1/k \leq \eta < 1$. Intuitively, η replaces the value $1/k$ in our algorithm. Consider the following modified online algorithm:

Fractional Caching algorithm: At time t, when page p_t is requested:

- Set the new variable: $x(p_t, r(p_t, t)) \leftarrow 0$. (It can only be increased in times $r(p_t, t) < t' < r(p_t, t) + 1$.)
- If the primal constraint corresponding to time t is satisfied, then do nothing.
- Otherwise: increase primal and dual variables, until the primal constraint corresponding to time t is satisfied, as follows:

  (1) Increase variable $y(t)$ continuously;
  for each variable $x(p, j)$ that appears in the (yet unsatisfied) primal constraint that corresponds to time t:

  (2) If $x(p, j) = 1$, then increase $z(p, j)$ at the same rate as $y(t)$.

  (3) If $x(p, j) = 0$ and
  $$\left(\sum_{t=t(p,j)+1}^{t(p,j+1)-1} y(t) \right) - z(p, j) = c_p,$$
  then set $x(p, j) \leftarrow \eta$.

  (4) If $\eta \leq x(p, j) < 1$, increase $x(p, j)$ according to the following function:
  $$\eta \cdot \exp\left(\frac{1}{c_p} \left[\left(\sum_{t=t(p,j)+1}^{t(p,j+1)-1} y(t) \right) - z(p, j) - c_p \right] \right).$$

It follows from the algorithm that $x(p, j) \leq 1$ implies that

$$\sum_{t=t(p,j)+1}^{t(p,j+1)-1} y(t) - z(p, j) \leq c_p(1 + \ln(1/\eta)),$$

and hence $y(t)$, scaled down by a factor of $(1 + \ln(1/\eta))$, is a feasible dual solution (to the dual of the linear program with only h pages). As before, we split the primal cost into two parts, C_1 and C_2, where C_1 is the contribution to the primal cost from due to the increase of variables $x(p,j)$ from 0 to η (line (3) of the algorithm), and C_2 is the contribution to the primal cost due to the increases of variables $x(p,j)$ according to the exponential function (line (4) of the algorithm).

We first observe that the argument for bounding C_2 follows along the exact same lines as the argument used for the case $h = k$. In particular, Δ, the derivative of the dual profit with respect to $y(t)$, is equal to $|B(t)| - h$ minus the number of variables $x(p,j)$ that have already reached the value of 1. Moreover, we have that

$$\frac{\mathrm{d}x(p,j)}{\mathrm{d}y(t)} = \frac{1}{c_p}x(p,j),$$

and hence the change in the primal cost is equal to the sum of the variables $x(p,j)$ that are at least η and strictly less than 1. Since the primal constraint is still unsatisfied, the sum of the variables $x(p,j)$ can be at most $|B(t)| - k$, which is at most $|B(t)| - h$ (as $k \geq h$), and hence the sum of the variables $x(p,j)$ that are strictly less than 1 is at most Δ. Thus, C_2 is bounded by the (scaled down) dual cost.

Bounding C_1 is also very similar to the case $h = k$. The only change is in the dual complementary slackness condition. In the modified online algorithm, we set the primal variables to η instead of $1/k$. Again, at time t let $B'(t)$ be the set of pages $p \in B(t)$ such that $x(p, r(p,t)) = 1$. In the dual solution, if $y(t)$ increases at time t, then:

$$\sum_{p \in B(t) \setminus (B'(t) \cup \{p_t\})} \tilde{x}(p, r(p,t)) \leq |B(t)| - h - |B'(t)|. \qquad (7.11)$$

Inequality (7.11) follows since there are $|B(t)| - 1 - |B'(t)|$ variables in the constraint corresponding to t. By definition for each p, $\tilde{x}(p, r(p,t)) \leq \eta = (k - h + 1)/k$. Thus, even if for all p, $\tilde{x}(p, r(p,t)) = (k - h + 1)/k$, the left-hand side adds up to $(|B(t)| - 1 - |B'(t)|)(k - h + 1)/k \leq |B(t)| - h - |B'(t)|$. The latter inequality holds since $|B(t)| - |B'(t)| \geq k + 1$, since otherwise the constraint at time t is already satisfied and the algorithm stops increasing the variable $y(t)$.

Thus, it follows that C_1 is bounded by the (scaled down) dual cost. Hence, $C_1 + C_2$ is at most $2(1 + \ln(1/\eta)) = 2(1 + \ln(k/(k - h + 1)))$ times the optimum solution, which implies that the modified online algorithm is $O(\log(k/(k - h + 1)))$-competitive.

7.2 Randomized Online Algorithm for Weighted Caching

In this section, we obtain a randomized algorithm by rounding online the fractional solution to an integral solution. A randomized algorithm defines a probability distribution on the various configurations (deterministic states) in each state of the algorithm. For the caching problem this corresponds to specifying the distribution on k-tuples of pages that are in the cache. Such a distribution induces another (simpler) distribution $x(p,t)$ on the pages, specifying the probability that a page is in the cache at time t. Clearly, this map is not a bijection. For example, the distribution $(1/2, 1/2, 1/2, 1/2)$ on four pages A, B, C, D could be induced by the distribution D_1 on two states (A, B) and (C, D), where each state occurs with probability $1/2$ each, or it can be induced by the distribution D_2 where each of six possible states (A, B), $(A, C), \ldots, (C, D)$ occur with probability $1/6$ each.

For the caching problem, the distribution on the pages can be viewed as a probability mass of k units distributed among the n pages, and the "move" of an algorithm simply corresponds to redistributing this mass among the pages. In this view, when an algorithm moves ϵ units of mass from page i, it incurs a cost of $\epsilon \cdot c_i$. We call this the *fractional view*, in contrast to working with the probability distribution on states which we call the *actual* view. We note that a fractional view can easily be obtained from a solution to a linear program (LP-caching), since the variables in the linear program indicate what fraction of a page is already evacuated from the cache. More formally, at time t the probability that page p is in the cache is simply $1 - x(p, r(p,t))$.

Our goal is to generate a randomized algorithm from a fractional view. The main difficulty in doing so is demonstrated in the following example. Consider the distribution $(1/2, 1/2, 1/2, 1/2)$ on pages A, B, C and D induced by the actual view, where cache states (A, B) and (C, D) each occur with probability $1/2$. Pages A and B have weight 1 and

pages C and D have a large weight M. Suppose the fractional algorithm moves $1/2$ unit of mass from page A to page B leading to the state of $(0, 1, 1/2, 1/2)$. In the fractional view, this algorithm incurs a cost of $1/2$. However, it is instructive to see that it is impossible to modify the actual distribution (to be consistent with the fractional distribution) without incurring a cost of $\Theta(M)$. In fact, the only actual distribution consistent with $(0, 1, 1/2, 1/2)$ is probability $1/2$ on state (B, C) and probability $1/2$ on state (B, D). Thus, from the previous cache state (C, D), either C or D must be moved to make room for B, which incurs cost $\Theta(M)$.

To get around this problem, we restrict our actual distributions to a certain subclass of distributions (e.g., in the scenario described above, we do not allow the distribution, in which states (A, B) and (C, D) have probability half each, to correspond to the distribution $(1/2, 1/2, 1/2, 1, 2)$). In particular, we show below how to maintain an online mapping from induced distributions to actual distributions, such that any fractional move with cost c is mapped to a move on actual distributions with cost at most $5c$.

We first round up the page fetching costs to their nearest power of 2 (increasing the competitive ratio by at most a factor of 2). Let $c_1 < c_2 < \cdots < c_\ell$ denote the rounded weights. A page belongs to class i if its rounded weight is c_i. We refer to an individual page as the jth page of class i. For convenience of analysis, throughout this section, we consider the (equivalent) cost version of the problem where we pay $c_i/2$ for both fetching and evicting a class i page.

Recall that in the fractional view of the problem, the algorithm maintains a distribution on the pages with total mass k. Any such distribution P is completely specified by $p_{ij} \in [0, 1]$ such that $\sum_i \sum_j p_{ij} = k$, where p_{ij} is the mass on the jth page of weight class i. Given two distributions P and P' on pages, let $C_f(P, P')$ denote the cheapest way to move from P to P', where it costs $c_i/2$ to move one unit of mass either into or out of a class i page. For those familiar, C_f is just the transshipment cost of flow from P to P' (we refer the reader to [38] for details about transshipment cost between distributions). Let $\delta_{ij} = p_{ij} - p'_{ij}$. Clearly, $C_f(P, P')$ is at least $\sum_i (c_i/2)(\sum_j |\delta_{ij}|)$ since at least $|\delta_{ij}|$ units of mass either needs to enter or leave page j of class i. Moreover, any

greedy algorithm that arbitrarily moves mass out of pages with excess ($\delta_{ij} > 0$) to those with a deficiency ($\delta_{ij} < 0$) has cost $\sum_i (c_i/2)(\sum_j |\delta_{ij}|)$ implying that $C_f(P, P') = \sum_i (c_i/2)(\sum_j |\delta_{ij}|)$.

A randomized algorithm on the other hand needs to work with a distribution on valid cache states. Given two distributions D and D' on the cache states, let $C(D, D')$ denote the cheapest way of moving from D to D' (by definition, this is the cost incurred by the randomized algorithm). Let $\Pi(D)$ denote the distribution induced on the pages by D. We say that P and D are *consistent* if $P = \Pi(D)$. Clearly, $C_f(\Pi(D), \Pi(D')$ is a lower bound on $C(D, D')$.

For the unweighted caching problem, Blum et al. [24] showed that given any P, P' and D such that $\Pi(D) = P$, there exists some D' such that $\Pi(D') = P'$ and $C(D, D') \leq 2C_f(P, P')$. Their procedure is the following. Suppose without loss of generality that P' is obtained from P by removing ϵ units of mass from page a and putting the mass on page b. Then, remove page a arbitrarily from ϵ measure of caches that contain a, and add page b to ϵ measure of caches that do not contain b. Now, some caches may have $k + 1$ pages (an excess) while some may have $k - 1$ pages (a hole). Arbitrarily match the caches with an excess to those with a hole (clearly, the measure of caches with excess is equal to those with a hole). Consider any matched pair; the cache with an excess must contain a page that does not lie in its matched cache, so we simply transfer this page. It can easily be verified that $C(D, D') \leq 2\epsilon$, while the fractional cost $C_f(P, P') = \epsilon$.

However, the situation for weighted caching is more involved. Recall our example that shows that there exist P, P' and D consistent with P, such that $C(D, D') \gg C_f(P, P')$ for every D' satisfying $P' = \Pi(D')$. Thus, we cannot work with any arbitrary D that is consistent with P, as in the unweighted case. Interestingly, we get around this problem by carefully restricting the space of distributions D that we are allowed to work with. Formally, we show the following.

Theorem 7.3. Let the costs c_i be such that $c_{i+1}/c_i \geq 2$ for $1 \leq i \leq \ell - 1$. There is a subclass $\mathcal{D}$ of distributions on cache states, along with a map T from $(\mathcal{D} \times \mathcal{P}) \to \mathcal{D}$ with the following property: Given any two distributions on pages P and P', and given any $D \in \mathcal{D}$ satisfying

$\Pi(D) = P$, we can obtain another distribution $D' = T(D, P')$ such that $\Pi(D') = P'$, $D' \in \mathcal{D}$ and $C(D, D') \leq 5C_f(P, P')$.

The theorem gives us the desired mapping between a distribution P on pages and a distribution D on cache states. Whenever the fractional algorithm moves from state P to P', the randomized algorithm moves from D to $D' = T(D, P')$. Since $\Pi(D') = P'$ and $D' \in \mathcal{D}$, the process can be applied repeatedly.

Proof. Let P be a distribution on pages with total mass k. Let $\mathcal{D}(P)$ denote the set of distributions $D \in \mathcal{D}$ that are consistent with P. Specifying $\mathcal{D}(P)$ for each P suffices to describe $\mathcal{D}$ completely. Each distribution $D \in \mathcal{D}$ is specified by associating a cache state $C(\alpha)$ with each real number α in the interval $[0, 1)$.

Let $k_i = \sum_j p_{ij}$ denote the mass on class i pages as determined by P. Consider the interval $I = [0, k]$, and imagine this interval partitioned into $I_1, \ldots, I_l$ where $I_1 = [0, k_1)$, $I_2 = [k_1, k_1 + k_2), \ldots$, $I_\ell = [k_1 + \ldots, k_{\ell-1}, k_1 + \cdots + k_\ell)$. Consider an $\alpha \in [0, 1)$. Let $T(\alpha)$ denote the set of real numbers $\{\alpha, 1 + \alpha, 2 + \alpha, \ldots, k - 1 + \alpha\}$. For every $D \in \mathcal{D}(P)$, the cache $C(\alpha)$ has n_i pages of c_i where $n_i = |T(\alpha) \cap I_i|$. By construction, each cache $C(\alpha)$ has either $\lfloor k_i \rfloor$ or $\lceil k_i \rceil$ pages of fetching cost c_i, and the expected number of pages of cost c_i is k_i. Consider any arbitrary way of filling the caches $C(\alpha)$, for $0 \leq \alpha < 1$, with pages such that: (i) no $C(\alpha)$ contains two identical pages and (ii) it is consistent with P (i.e., the probability measure of caches that contain page j of class i is exactly p_{ij}). Such a filling always exists since, for example, we can put the first page of class 1 in $C(\alpha)$ corresponding to $\alpha = [0, p_{11})$, the second page of class 1 in $C(\alpha)$ corresponding to $\alpha = [p_{11}, p_{11} + p_{12})$ (where the range of α is considered modulo 1) and so on. Any way of filling $C(\alpha)$ that satisfies the properties above is a valid element $D \in \mathcal{D}(P)$.

We now describe the transformation T. Suppose we are given some $D \in \mathcal{D}(P)$, and the fractional algorithm changes state from P to P'. By separating the pages for which $p'_{ij} > p_{ij}$ and those for which $p'_{ij} < p_{ij}$ and arbitrarily matching the increases in mass with decreases, we can decompose the move P to P' into

the sequence $P = P_0 \to P_1 \to P_2 \to \cdots \to P'$ such that $C_f(P, P') = \sum_{i \geq 0} C_f(P_i, P_{i+1})$ and each move P_i to P_{i+1} is an *exchange* where some infinitesimally small ϵ units of mass is moved from some page p_a to some page p_b. Thus, it suffices to prove the theorem for such exchanges $P \to P'$. Let i be the weight class of page a, and j be that of page b. For this move, the fractional algorithm pays $\epsilon(c_i + c_j)/2$.

We now describe and analyze the move $D \to D'$. We divide the cost into two parts. One due to cache size changes, and the second due to the change in the composition of the cache. We first consider the simpler case when $i = j$. Here, the quantities $k_1, \ldots, k_\ell$ and the intervals $I_1, \ldots, I_\ell$ associated with P remain unchanged, and hence the structure of $C(\alpha)$ remains unchanged. We essentially apply the argument of Blum et al. [24] to class i pages. The only difference is that we need to verify that their argument works even when caches contain either $\lceil k_i \rceil$ or $\lfloor k_i \rfloor$ class i pages (in [24] all caches have the same size). We arbitrarily remove page a from an ϵ measure of caches that contain a, and arbitrarily add b to an ϵ measure of caches that do not contain b. We say that a cache has a hole if it has one fewer page than it is supposed to, and it has an excess if it has one extra page than it is supposed to. Any cache with a hole has size either $\lfloor k_i \rfloor - 1$ or $\lfloor k_i \rfloor$, and every cache with excess has size either $\lceil k_i \rceil$ or $\lceil k_i \rceil + 1$, and hence is strictly larger. We arbitrarily pair up the caches with a hole to those with excesses, and transfer some page from the larger cache that does not lie in the smaller cache. The cost incurred is at most $2\epsilon c_i/2 + 2\epsilon c_i/2 = 2\epsilon c_i$.

We now consider the case when $i < j$ (the case when $i > j$ is analogous). Consider the intervals $I_1, \ldots, I_\ell$. When we move from P to P' the right boundary of I_i shifts ϵ units to the left, the intervals $I_{i+1}, \ldots, I_{j-1}$ shift to the left by ϵ units, and finally, the left boundary of I_j shifts left by ϵ and its right boundary stays fixed.

We break the analysis into two parts. We first consider the classes h for $i < h < j$. For each such h at most ϵ fraction of caches $C(\alpha)$ must lose a page of cost c_h (as their quota for class h shrinks from $\lceil k_h \rceil$ to $\lfloor k_h \rfloor$ and similarly, at most ϵ fraction of caches must gain a page). Moreover, the fraction of caches that must lose a cost c_h page is exactly equal to the fraction that must gain such a page. We arbitrarily pair these caches. As any cache that must lose a page is strictly larger than

a cache that must gain one, for every matched pair of caches, there is some page in the larger cache that does not lie in the smaller cache and hence can be transferred to it. The movement cost incurred per class is at most $2\epsilon(c_h/2)$, and hence the total contribution due to such classes h is $\sum_{i<h<j} \epsilon c_h \leq \epsilon(c_i + c_j)$, as consecutive weights differ by a factor of at least 2.

Finally, we consider the case when $h = i$ (the argument for $h = j$ is analogous). Without loss of generality we assume that $\lceil k_i \rceil = \lceil k_i - \epsilon \rceil$ (otherwise we can split ϵ into at most two parts ϵ_1, ϵ_2, and apply the argument separately). Consider the caches $C(\alpha)$ that are supposed to lose a cost c_i page (because k_i becomes $k_i - \epsilon$). We say that these caches have an excess, and note that they all contain exactly $\lceil k_i \rceil$ class i pages. Next, we arbitrarily choose ϵ measure of caches that contain a, and remove a from them. These caches have a hole, and strictly fewer class i pages than caches with excess (a cache with a hole has either $\lfloor k_i \rfloor$ or $\lfloor k_i \rfloor - 1$ pages). We arbitrarily pair caches with an excess to caches with a hole, and transfer some page from the larger cache that does not lie in the smaller cache. The cost incurred is at most $3\epsilon c_i/2$. By an identical argument for class j, the cost incurred is at most $3\epsilon c_j/2$.

The distribution D' obtained satisfies all the conditions required for it to lie in the set $\mathcal{D}(P')$. Moreover, the total cost incurred in moving from D to D' is $5\epsilon(c_i + c_j)/2$ which is at most five times the fractional cost. □

7.3 The Generalized Caching Problem

In this section, we extend the main ideas presented earlier for weighted caching and design an algorithm for the generalized caching problem. In the generalized caching problem, there is a cache of size k and n pages of sizes $w_1 \leq w_2 \leq \cdots \leq w_n$, belonging to $\in [1, k]$. It is not assumed that page sizes are integral and k can be viewed as the ratio between the cache size and the smallest page size. For any subset S of pages, let $W(S) = \sum_{p \in S} w_p$ be the sum of the sizes of the pages in S. Page p has a fetching cost of c_p. With this terminology, in the fault model $c_p = 1$ for each page p, in the bit model $c_p = w_p$ for each page p, and in the general model the values of c_p and w_p are arbitrary.

7.3.1 LP Formulation for Generalized Caching

The formulation that we used for the weighted caching can easily be extended to the case of generalized caching. This obvious extension gives us the following integer formulation for the problem:

$$\min \sum_{p=1}^{n} \sum_{j=1}^{r(p,t)} c_p \cdot x(p,j).$$

For any time t:

$$\sum_{p \in B(t) \setminus \{p_t\}} w_p x(p, r(p,t)) \geq W(B(t)) - k.$$

For any p, j:

$$x(p,j) \in \{0,1\}.$$

In a fractional solution, we relax $x(p,j)$ to take any value between 0 and 1. However, there is a fundamental problem with this relaxation, as it can have an integrality gap of $\Omega(k)$, and therefore is not suitable for our purposes. For example, suppose the cache size is $k = 2\ell - 1$, and there are two pages of size ℓ, requested alternately. Only one page can be in the cache at any time and hence there is a cache miss in each request. A fractional solution, on the other hand, can keep almost one unit of each page and then it only needs to fetch an $O(1/k)$ fraction of a page in each request.

To get around this problem, we use an idea introduced by Carr et al. [36] of adding exponentially many *knapsack cover* inequalities. These constraints are redundant in the integer program, but they dramatically reduce the integrality gap of the LP relaxation. There are two main ideas. First, consider a subset of pages $S \subset B(t)$ such that $p_t \in S$ and $W(S) > k$. The pages in $S \setminus \{p_t\}$ can occupy at most $k - w_{p_t}$ units in the cache at time t. Thus, at least $W(S) - w_{p_t} - (k - w_{p_t}) = W(S) - k$ cumulative size of pages in $S \setminus \{p_t\}$ must be absent from the cache. Hence, we can add the constraint $\sum_{p \in S \setminus \{p_t\}} w_p x(p, r(p,t)) \geq W(S) - k$ for each such set S at time t. The second idea is that for each such constraint, we can truncate the size of a page to be equal to the right-hand side of the constraint, i.e., we have

$\sum_{p \in S \setminus \{p_t\}} \min(W(S) - k, w_p) x(p, r(p,t)) \geq W(S) - k$. Clearly, truncating the size has no effect on the integer program. Our linear program is as follows:

$$\min \sum_{p=1}^{n} \sum_{j=1}^{r(p,t)} c_p \cdot x(p,j)$$

For any time t and any set of requested pages $S \subseteq B(t)$ such that $p_t \in S$ and $W(S) > k$:

$$\sum_{p \in S \setminus \{p_t\}} \min\{W(S) - k, w_p\} x(p, r(p,t)) \geq W(S) - k. \tag{7.12}$$

For any p, j:

$$0 \leq x(p,j) \leq 1. \tag{7.13}$$

We now note a simple observation about knapsack cover inequalities that will be quite useful.

Observation 7.4. Given a fractional solution x, if a knapsack cover inequality is violated for a set S at time t, then it is also violated for the set $S' = S \setminus \{p : x(p, r(p,t)) = 1\}$, obtained by omitting pages which are already completely evicted from the cache.

Proof. Suppose that inequality (7.12) is violated for some S and $x(p, r(p,t)) = 1$ for $p \in S$. First, it must be the case that $\min(W(S) - k, w_p) < W(S) - k$, otherwise (7.12) is trivially satisfied. Suppose we delete p from S. The right-hand side decreases by exactly w_p. The left-hand side decreases by w_p and possibly more since the term $\min(W(S) - k, w_{p'})$ may decrease for pages $p' \in S$. Thus, inequality (7.12) is also violated for $S \setminus \{p\}$. The claim follows by repeatedly applying the argument. □

Observation 7.4 implies that in any feasible solution to the constraints given by (7.12), it does not "help" having $x(p,j) > 1$. Hence, it can be assumed that $x(p,j) \leq 1$ without loss of generality, and we can

drop the upper bounds on $x(p,j)$, simplifying the LP formulation to:

$$\min \sum_{p=1}^{n} \sum_{j=1}^{r(p,t)} c_p \cdot x(p,j) \quad \text{(LP-caching)}$$

For any time t and any set of requested pages $S \subseteq B(t)$ such that $p_t \in S$ and $W(S) > k$:

$$\sum_{p \in S \setminus \{p_t\}} \min\{W(S) - k, w_p\} x(p, r(p,t)) \geq W(S) - k. \tag{7.14}$$

For any p, j:

$$0 \leq x(p,j). \tag{7.15}$$

In the dual program, there is a variable $y(t,S)$ for each time t and set $S \subseteq B(t)$ such that $p_t \in S$ and $W(S) > k$. The dual program is as follows:

$$\max \sum_{t} \sum_{S \subseteq B(t), p_t \in S} (W(S) - k)\, y(t,S).$$

For each page p and the jth time it is requested:

$$\sum_{t=t(p,j)+1}^{t(p,j+1)-1} \sum_{S \mid p \in S} \min\{W(S) - k, w_p\} y(t,S) \leq c_p. \tag{7.16}$$

We will sometimes denote $\min\{W(S) - k, w_p\}$ by $\tilde{w}_p^S$.

7.3.2 A Fractional Primal–Dual Algorithm

Our online caching algorithm produces fractional primal and dual solutions to LP-caching and it is very similar to the weighted caching case (where pages have uniform size). In the online case, the constraints of LP-caching are revealed in sets — at any time t exponentially many new linear knapsack-cover constraints appear and the goal is to produce a feasible assignment to the (primal) variables that satisfies all the constraints. Since there are exponentially many constraints, polynomial running time is not obvious; however, we later show that our online algorithm can be made to run in polynomial time.

Upon arrival of a new set of constraints at time t, if all constraints are already satisfied, then the algorithm does nothing. Otherwise, the algorithm needs to satisfy all the current constraints by increasing some of the primal variables. We call a set S minimal if $x(p, r(p,t)) < 1$ for each $p \in S$. By Observation 7.4, it suffices to consider primal constraints corresponding to minimal sets. Satisfying all the constraints at time t guarantees that there is enough space (fractionally) in the cache to fetch the new page.

To this end, the algorithm arbitrarily picks an unsatisfied primal constraint corresponding to some minimal set S and starts increasing continuously its corresponding dual variable $y(t,S)$. This, in turn, tightens some of the dual constraints corresponding to primal variables $x(p,j)$ whose current value is 0. Whenever such an event happens, the value of $x(p,j)$ is increased from its initial setting of 0 to $1/k$. Meanwhile, variables $x(p,j)$ which are already set to $1/k$ are increased (continuously) according to an exponential function of the new dual variable $y(t,S)$. When variable $x(p,j)$ reaches 1, the set S is no longer minimal, and page p is dropped from S. As a result, from this time on, the value of $x(p,j)$ remains 1. When this primal constraint is satisfied the algorithm continues on to the next infeasible primal constraint.

Since there are exponentially many primal constraints in each iteration this process may not be polynomial. However, the rounding process we design in Section 7.4 does not need the solution to satisfy all primal constraints. Specifically, for each model we show that there exists a (different) value $\gamma > 1$ such that the algorithm needs to guarantee that at time t the primal constraint of the set $S = \{p \mid x(p, r(p,t)) < 1/\gamma\}$ is satisfied. Thus, the algorithm may actually consider only that set.[1] Fortunately, the requirement of the online primal–dual framework that variables can only increase monotonically makes this task simple. In particular, as the primal variables increase, some pages reach $1/\gamma$ and "leave" the set S. The algorithm then tries to satisfy the set S' that contains the rest of the pages. Since pages can only leave S, this process may continue for at most n rounds. For simplicity, we describe the

[1] In general, for knapsack cover constraints in an offline setting, all possible subsets may be needed since it is not clear *a priori* which set S will have this property, nor can it be expressed as a linear or even a convex program. See [36] for more details.

algorithm that satisfies all the constraints. The algorithm is presented in a continuous fashion, but it can easily be implemented in a discrete fashion. The algorithm is the following:

Fractional caching algorithm: At time t, when page p_t is requested:

- Set the new variable: $x(p_t, r(p_t,t)) \leftarrow 0$. (It can only be increased at times $r(p_t,t) < t' < r(p_t,t)+1$.)
- Until *all* the primal constraint corresponding to time t are satisfied do the following. Assume that the primal constraint of minimal set S is not satisfied:
  (1) Increase variable $y(t,S)$ continuously; for each variable $x(p,j)$ such that $p \in S \setminus \{p_t\}$:
  (2) If $x(p,j) = 1$, then remove p from S, i.e. $S \leftarrow S \setminus \{p\}$.
  (3) If $x(p,j) = 0$ and $\sum_{t=t(p,j)+1}^{t(p,j+1)-1} \sum_{S:p\in S} \tilde{w}_p^S y(t,S) = c_p$, then $x(p,j) \leftarrow 1/k$.
  (4) If $1/k \le x(p,j) < 1$, increase $x(p,j)$ according to the following function:

$$\frac{1}{k}\exp\left(\frac{1}{c_p}\left[\left(\sum_{t=t(p,j)+1}^{t(p,j+1)-1}\sum_{S:p\in S}\tilde{w}_p^S y(t,S)\right) - c_p\right]\right),$$

  where $\tilde{w}_p^S$ denotes $\min\{W(S)-k, w_p\}$.

Theorem 7.5. The algorithm is $O(\log k)$-competitive.

Proof. The proof of the theorem is along the same lines as the proof of Theorem 7.1. First, we note that the primal solution generated by the algorithm is feasible. This follows since, in each iteration, the variables $x(p,j)$ are increased until all new primal constraints are satisfied. Also, each variable $x(p,j)$ is never increased to be greater than 1.

Next, we show that the dual solution that we generate is feasible up to an $O(\log k)$ factor. Whenever $x(p,j)$ reaches 1, the variables

$y(t,S)$ for sets S containing p do not increase anymore, and hence the value of $x(p,j)$ does not change any more. Thus, for the dual constraint corresponding to page p and the jth time it is requested, we get that:

$$x(p,j) = \frac{1}{k}\exp\left(\frac{1}{c_p}\left[\left(\sum_{t=t(p,j)+1}^{t(p,j+1)-1}\sum_{S:p\in S}\tilde{w}_p^S y(t,S)\right) - c_p\right]\right) \leq 1$$

where $\tilde{w}_p^S = \min\{W(S)-k, w_p\}$. Simplifying, we get that:

$$\sum_{t=t(p,j)+1}^{t(p,j+1)-1}\sum_{S \mid p\in S}\min\{W(S)-k,w_p\}y(t,S) \leq c_p(1+\ln k).$$

Thus, the dual solution can be made feasible by scaling it down by a factor of $(1+\ln k)$. We now prove that the primal cost is at most twice the dual profit, which means that the primal solution produced is $O(\log k)$-competitive.

We partition the primal cost into two parts, C_1 and C_2. Let C_1 be the contribution to the primal cost from step (3) of the algorithm, due to the increase of variables $x(p,j)$ from 0 to $1/k$. Let C_2 be the contribution to the primal cost from step (4) of the algorithm, due to the incremental increases of the variable $x(p,j)$ according to the exponential function.

Bounding C_1: Let $\tilde{x}(p,j) = \min(x(p,j),1/k)$. We bound the term $\sum_{p=1}^{n}\sum_{j=1}^{r(p,t)} c_p\tilde{x}(p,j)$. To do this, we need two observations. First, it follows from the algorithm that if $x(p,j)>0$, and equivalently if $\tilde{x}(p,j)>0$, then:

$$\sum_{t=t(p,j)+1}^{t(p,j+1)-1}\sum_{S \mid p\in S}\min\{W(S)-k,w_p\}y(t,S) \geq c_p. \tag{7.17}$$

We shall refer to (7.17) as primal complementary slackness. Next, in the dual solution, if $y(t,S)>0$, then:

$$\sum_{p\in S\setminus\{p_t\}}\min\{W(S)-k,w_p\}\tilde{x}(p,r(p,t)) \leq W(S)-k. \tag{7.18}$$

We shall refer to (7.18) as dual complementary slackness. To see why (7.18) holds, consider the following two cases depending on whether

$|S| \geq k+1$ or not. Recall that $\tilde{x}(p, r(p,t)) \leq 1/k$ for all pages. If $|S| \geq k+1$ then $W(S) \geq k+1$ and so:

$$\begin{aligned}\sum_{p \in S \setminus \{p_t\}} \frac{1}{k} \min\{W(S)-k, w_p\} &\leq \frac{1}{k} \cdot \sum_{p \in S \setminus \{p_t\}} w_p \\ &= \frac{W(S)-w(p_t)}{k} \leq \frac{W(S)-1}{k} \leq W(S)-k.\end{aligned}$$

If $|S| \leq k+1$, then:

$$\begin{aligned}\sum_{p \in S \setminus \{p_t\}} \frac{1}{k} \min\{W(S)-k, w_p\} &\leq \frac{1}{k} \sum_{p \in S \setminus \{p_t\}} (W(S)-k) \\ &\leq \frac{k}{k}(W(S)-k) = W(S)-k.\end{aligned}$$

The last inequality follows since $W(S) \geq k$. The primal and dual complementary slackness conditions imply the following:

$$\sum_{p=1}^{n} \sum_{j=1}^{r(p,t)} c_p \tilde{x}(p,j) \tag{7.19}$$

$$\leq \sum_{p=1}^{n} \sum_{j=1}^{r(p,t)} \left(\sum_{t=t(p,j)+1}^{t(p,j+1)-1} \sum_{S \mid p \in S} \tilde{w}_p^S y(t,S) \right) \tilde{x}(p,j) \tag{7.20}$$

$$= \sum_{t} \sum_{S \subseteq B(t), p_t \in S} \left(\sum_{p \in S \setminus \{p_t\}} \tilde{w}_p^S \tilde{x}(p, r(p,t)) \right) y(t,S) \tag{7.21}$$

$$\leq \sum_{t} \sum_{S \subseteq B(t), p_t \in S} (W(S)-k)\, y(t,S). \tag{7.22}$$

Inequality (7.20) follows from inequality (7.17), Equality (7.21) follows by changing the order of summation, and inequality (7.22) follows from inequality (7.18). Thus, C_1 is at most the profit of a *feasible* dual solution multiplied by $(1 + \ln k)$.

Bounding C_2: We bound the derivative of the primal cost of variables $x(p,j)$ in step (4) by the derivative of the dual profit accrued in the same round. Variables $x(p,j)$ that have already reached the value of 1 do not

contribute anymore to the primal cost. The derivative of a variable $x(p,j)$, $1/k \leq x(p,j) < 1$, as a function of $y(t)$ is

$$\frac{\mathrm{d}x(p,j)}{\mathrm{d}y(t,S)} = \frac{\min\{W(S)-k, w_p\}}{c_p} x(p,j). \tag{7.23}$$

Therefore, the derivative of the primal is at most:

$$\begin{aligned}\frac{\mathrm{d}X}{\mathrm{d}y(t,S)} &= \sum_{p \in S \setminus \{p_t\} : x(p,r(p,t)) < 1} \tilde{w}_p^S x(p, r(p,t)) \\ &\leq W(S) - k = \frac{\mathrm{d}Y}{\mathrm{d}y(t,S)}.\end{aligned}$$

The inequality in the second step above follows since the primal constraint of the set S is still satisfied. Thus, C_2 is at most the profit of a feasible dual solution multiplied by $(1 + \ln k)$.

Completing the analysis: It follows that $C_1 + C_2$ is at most twice the profit of a *feasible* dual solution multiplied by $(1 + \ln k)$. Note that the profit of any dual feasible solution is always a lower bound on the optimal solution. Therefore, we conclude by weak duality that the algorithm is $O(\log k)$-competitive. □

7.4 Rounding the Fractional Solution Online

In this section, we show how to obtain a randomized (integral) online algorithm from the fractional solution generated in the previous section for LP-caching. The ideas here generalize those used earlier in the simpler weighted caching case. For convenience of analysis, throughout this section we consider the (equivalent) cost version of the problem where we pay c_p for both fetching and evicting a page p. This assumption can change the cost of the fractional solution by at most a factor of two. At any point of time, the LP solution $x_1, \ldots, x_n$ (for LP-caching), where we denote by $x_p \triangleq x(p, r(p,t))$, specifies the probability that each of the pages is absent from the cache. However, in order to obtain an actual randomized algorithm we need to specify a probability distribution over the various cache states that is consistent with the LP solution. That is, we need to simulate the moves of the LP over the set of pages by consistent moves over the actual cache states. We adopt the following approach to do this simulation.

Let $\gamma \geq 1$ be a parameter and set $y_p = \min(\gamma x_p, 1)$. Let μ be a distribution on subsets of pages. We say that μ is consistent with y (or γ-consistent with x) if μ induces the distribution y on the page set. That is,

$$\forall p : \sum_D A_p^D \cdot \mu(D) = y_p, \tag{7.24}$$

where, for a set of pages D, $A_p^D = 1$ if $p \in D$ and 0 otherwise. We will view μ as a distribution over the complement of the cache states. To be a meaningful simulation, it suffices to require the following:

(1) *Size property*: For any set D with $\mu(D) > 0$, the sum of the sizes of the pages in D is at least $W(B(t)) - k$. That is, D corresponds to the complement of a valid cache.
(2) *Bounded cost property*: If y changes to y' while incurring a fractional cost of d, the distribution μ can be changed to another distribution μ' which is consistent with y', while incurring a (possibly amortized) cost of at most βd, where $\beta > 0$.

It is easy to see that if x_p changes by ϵ, then y_p changes by at most $\gamma\epsilon$. Hence, given a fractional algorithm with competitive ratio c, the existence of a simulation with the above properties implies an actual randomized online algorithm with competitive ratio $\gamma\beta c$. We provide three different simulation procedures for the bit model, general model, and the fault model. These are organized in increasing order of complexity.

7.4.1 The Bit Model

In this section, we will show how to obtain an $O(\log k)$-competitive randomized algorithm for the generalized caching problem in the bit model. Let $U \triangleq \lfloor \log_2 k \rfloor$. For $i = 0$ to U, we define the size class $S(i)$ to be the set of pages of sizes between 2^i and less than size 2^{i+1}. Formally, $S(i) = \{p \mid 2^i \leq w_p < 2^{i+1}\}$. Let $x_1, \ldots, x_n$ be the LP solution at the current time step. Recall that it satisfies the knapsack cover inequalities for all subsets. For each page p let $y_p = \min\{1, 3x_p\}$ (i.e., $\gamma = 3$).

Definition 7.1 (Balanced subsets). We say that a subset of pages D is balanced with respect to y if:

(1) If $y_p = 1$ then p is evicted in all cache states, i.e., $A_p^D = 1$ for all D with $\mu(D) > 0$.
(2) The following holds for all $0 \leq j \leq U$:

$$\left\lfloor \sum_{i=j}^{U} \sum_{p \in S(i)} y_p \right\rfloor \leq \sum_{i=j}^{U} \sum_{p \in S(i)} A_p^D \leq \left\lceil \sum_{i=j}^{U} \sum_{p \in S(i)} y_p \right\rceil . \quad (7.25)$$

We first show that the size property follows from the requirement that sets are balanced.

Lemma 7.6. Let x and y be defined as above. Then, for any subset D which is balanced with respect to y, the sum of the sizes of all the pages in D is at least $W(B(t)) - k$.

We first prove a simple mathematical claim.

Claim 7.7. Let $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ be two sequences of non-negative real numbers and let $0 = a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_n$ be a non-decreasing sequence of positive numbers. If for every $1 \leq j \leq n$: $\sum_{i=j}^{n} x_i \geq -1 + (\sum_{i=j}^{n} y_i)$, then: $\sum_{i=1}^{n} a_i x_i \geq -a_n + \sum_{i=1}^{n} a_i y_i$.

Proof. For every j, $1 \leq j \leq n$, multiply the jth inequality by $(a_j - a_{j-1})$ (which is non-negative), yielding:

$$(a_j - a_{j-1}) \sum_{i=j}^{n} x_i \geq -(a_j - a_{j-1}) + (a_j - a_{j-1}) \sum_{i=j}^{n} y_i.$$

Summing up over all the inequalities yields the desired result. □

Proof [Lemma 7.6]. For the proof it suffices to use the left-hand side of condition (7.25) (i.e., the lower bound). Let $S' \subseteq S$ be the set of pages with $y_p < 1$, and let $S'(i) = S' \cap S(i)$ be the class i pages in S'.

Since $A_p^D = 1$ whenever $y_p = 1$, condition (7.25) implies that for every $0 \le j \le U$:

$$\sum_{i=j}^{U} \sum_{p \in S'(i)} A_p^D \ge \left\lfloor \sum_{i=j}^{U} \sum_{p \in S'(i)} y_p \right\rfloor \ge \left(\sum_{i=j}^{U} \sum_{p \in S'(i)} y_p \right) - 1. \quad (7.26)$$

The sum of the sizes of the pages in D is $\sum_{p \in S} w_p A_p^D$. Since $A_p^D = 1$ for $p \in S \setminus S'$, it suffices to show that $\sum_{p \in S'} w_p A_p^D \ge W(S') - k$ for the proof. Consider the following:

$$\begin{aligned}
\sum_{p \in S'} w_p A_p^D &\ge \sum_{p \in S'} \min\{w_p, W(S') - k\} A_p^D \\
&= \sum_{i=0}^{U} \sum_{p \in S'(i)} \min\{w_p, W(S') - k\} A_p^D \\
&\ge \frac{1}{2} \sum_{i=0}^{U} \sum_{p \in S'(i)} \min\{2w_p, W(S') - k\} A_p^D \\
&\ge \frac{1}{2} \sum_{i=0}^{U} \min\{2^{i+1}, W(S') - k\} \sum_{p \in S'(i)} A_p^D \quad (7.27) \\
&\ge -\frac{1}{2} \min\{2^{U+1}, W(S') - k\} \\
&\quad + \frac{1}{2} \sum_{i=0}^{U} \min\{2^{i+1}, W(S') - k\} \sum_{p \in S'(i)} y_p \quad (7.28) \\
&\ge -\frac{1}{2} (W(S') - k) \\
&\quad + \frac{1}{2} \sum_{i=0}^{U} \sum_{p \in S'(i)} \min\{w_p, W(S') - k\} y_p \quad (7.29) \\
&\ge -\frac{1}{2} (W(S') - k) + \frac{3}{2} (W(S') - k) \ge W(S') - k. \quad (7.30)
\end{aligned}$$

Here, inequality (7.27) follows since $w_p \ge 2^i$ for each $p \in S'(i)$. Inequality (7.28) follows by applying Claim 7.7 with $a_i = \min\{2^{i+1}, W(S') - k\}$, $x_i = \sum_{p \in S'(i)} A_p^D$ and $y_i = \sum_{p \in S'(i)} y_p$, and observing that (7.26)

implies that the conditions of the claim are satisfied. Inequality (7.29) follows since $w_p < 2^{i+1}$, and finally inequality (7.30) follows since by the LP knapsack constraints, and the fact that $y_p = 3x_p$ for each $p \in S'$:

$$\sum_{i=0}^{U} \sum_{p \in S'(i)} \min\{w_p, W(S') - k\} y_p$$
$$= 3 \sum_{p \in S'} \min\{w_p, W(S') - k\} x_p \geq 3(W(S') - k). \qquad \square$$

We show how to maintain the bounded cost property using both left- and right-hand sides of condition (7.25).

Lemma 7.8. Let μ be any distribution on balanced sets that is consistent with y. Then the cost property holds with $\beta = 10$. That is, if y changes to y' while incurring a fractional cost of d, then the distribution μ can be modified to another distribution μ' over balanced sets such that μ' is consistent with y' and the cost incurred while modifying μ to μ' is at most $10d$.

Proof. By considering each page separately, it suffices to show that the property holds whenever y_p increases or decreases for some page p. Assume first that the weight y_p of page p for $p \in S(i)$ is increased by ϵ. The argument when y_p is decreased is analogous. Page p belongs to $S(i)$, and so $w_p \geq 2^i$. Thus, the fractional cost is at least $\epsilon 2^i$.

We construct μ' as follows. To ensure the consistency with y', i.e., Equation (7.24), we add page p to ϵ measure of the sets D that do not contain p, incurring a cost of at most $2^{i+1}\epsilon$ (in the bit model). However, condition (7.25) for classes $j \leq i$ may now be violated. We iteratively fix condition (7.25) starting with class i. Consider class i. Let $s = \lceil \sum_{j=i}^{U} \sum_{p \in S(j)} y_p \rceil$ and suppose first that $\lceil \sum_{j=i}^{U} \sum_{p \in S(j)} y'_p \rceil$ remains equal to s. Then in μ', let ϵ' be the measure of sets that have $s+1$ pages in classes i or higher. Note that $\epsilon' \leq \epsilon$. Consider the sets with $s-1$ pages in classes i or higher and arbitrarily choose ϵ' measure of these (this is possible since $s = \lceil \sum_{j=i}^{U} \sum_{p \in S(j)} y'_p \rceil$). Arbitrarily pair the sets with $s+1$ pages to those with $s-1$ pages. Consider any

pair of sets (D, D'). Since μ' satisfies condition (7.25) for class $i+1$, the number of pages in D and D' that lie in classes $i+1$ or higher differ by at most 1. Hence, $D \setminus D'$ contains some class i page. We move this page from D to D'. Note that (7.25) is satisfied for i after this procedure. Now, consider the case when $\lceil \sum_{j=i}^{U} \sum_{p \in S(j)} y'_p \rceil$ increases to $s+1$. Note that in this case, condition (7.25) is violated for class i for at most $\epsilon' \leq \epsilon$ of the sets that have precisely $s-1$ pages in classes i or higher. We arbitrarily pair the classes with $s-1$ pages to those with $s+1$ pages and apply the argument above. The total cost incurred in this step is at most $(2\epsilon') \cdot 2^{i+1} \leq 2^{i+2}\epsilon$.

After applying the above procedure to fix class i, condition (7.25) might be violated for class $i-1$ for at most ϵ measure of sets. We apply the matching procedure sequentially to $i-1$ and lower classes incurring an additional cost of $\sum_{j=0}^{i-1} 2\epsilon \cdot 2^{j+1} < 4\epsilon 2^i$. Thus, the total cost incurred is at most $10\epsilon 2^i$. □

Theorem 7.9. There is an $O(\log k)$-competitive algorithm for the generalized caching problem in the bit model.

7.4.2 The General Cost Model

In this section, we study the general cost model and show how to obtain an $O(\log^2 k)$-competitive randomized caching algorithm for this model. Let $U \triangleq \lfloor \log_2 k \rfloor$. Let $C = \lfloor \log_2 C_{\max} \rfloor$. For $i = 0$ to U, and $j = 0$ to C, we define $S(i,j)$ to be the set of pages of sizes at least 2^i and less than 2^{i+1}, and fetching cost between 2^j and less than 2^{j+1}. Formally, $S(i,j) = \{p \mid 2^i \leq w_p < 2^{i+1} \text{ and } 2^j \leq c_p < 2^{j+1}\}$. Let $x_1, \ldots, x_n$ be the LP solution at the current time step that satisfies the knapsack cover inequalities for all subsets. Let $\gamma = U + 3$. Thus, for each page p, $y_p = \min\{1, (U+3) \cdot x_p\} = O(\log k) \cdot x_p$.

Definition 7.2. A set D of pages is balanced with respect to y if the following two conditions hold:

(1) If $y_p = 1$ then p is evicted in all cache states, i.e., $A_p^D = 1$ for all D with $\mu(D) > 0$.

(2) For each size class $0 \leq i \leq U$, it holds that for each $0 \leq j \leq \lfloor \log C_{\max} \rfloor$:

$$\left\lfloor \sum_{z=j}^{C} \sum_{p \in S(i,z)} y_p \right\rfloor \leq \sum_{z=j}^{C} \sum_{p \in S(i,z)} A_p^D \leq \left\lceil \sum_{z=j}^{C} \sum_{p \in S(i,z)} y_p \right\rceil . \quad (7.31)$$

We first show that the size property follows from the requirement that the sets are balanced.

Lemma 7.10. Let x and y be defined as above. Then, for any subset D that is balanced with respect to y, the sum of the sizes of all the pages in D is at least $W(B(t)) - k$.

Proof. For the proof it suffices to use the left-hand side of condition (7.31) (i.e., the lower bound). Let S' denote the subset of pages with $y_p < 1$. As $y_p = 1$ whenever $A_p^D = 1$, it suffices to show that $\sum_{p \in S'} w_p A_p^D \geq W(S') - k$. Moreover, condition (7.31) implies that for any $0 \leq i \leq U$:

$$\sum_{z=0}^{C} \sum_{p \in S'(i,z)} A_p^D \geq \lfloor \sum_{z=0}^{C} \sum_{p \in S'(i,z)} y_p \rfloor \geq -1 + \sum_{z=0}^{C} \sum_{p \in S'(i,z)} y_p. \quad (7.32)$$

Thus, the total size of pages from S' that are in D can be lower bounded as follows:

$$\begin{aligned}
\sum_{p \in S'} w_p A_p^D &\geq \sum_{p \in S'} \min\{w_p, W(S') - k\} A_p^D \\
&= \sum_{i=0}^{U} \sum_{j=0}^{C} \sum_{p \in S'(i,j)} \min\{w_p, W(S') - k\} A_p^D \\
&\geq \frac{1}{2} \sum_{i=0}^{U} \sum_{j=0}^{C} \sum_{p \in S'(i,j)} \min\{2w_p, W(S') - k\} A_p^D \\
&\geq \frac{1}{2} \sum_{i=0}^{U} \min\{2^{i+1}, W(S') - k\} \sum_{j=0}^{C} \sum_{p \in S'(i,j)} A_p^D \qquad (7.33)
\end{aligned}$$

$$\geq \frac{1}{2}\sum_{i=0}^{U}\min\{2^{i+1}, W(S')-k\}\left(-1+\sum_{j=0}^{C}\sum_{p\in S'(i,j)} y_p\right) \tag{7.34}$$

$$\geq -\frac{U+1}{2}(W(S')-k)$$

$$+\frac{1}{2}\sum_{i=0}^{U}\sum_{j=0}^{C}\sum_{p\in S'(i,j)}\min\{w_p, W(S')-k\}y_p \tag{7.35}$$

$$\geq -\frac{U+1}{2}(W(S')-k)+\frac{U+3}{2}(W(S')-k)$$

$$= W(S')-k. \tag{7.36}$$

Inequality (7.33) follows since $w_p \geq 2^i$ for each page $p \in S'(i,j)$, and inequality (7.34) follows from (7.32). Inequality (7.35) follows since $w_p \leq 2^{i+1}$ for each page $p \in S'(i,j)$. Finally, inequality (7.36) follows from the knapsack constraints:

$$\sum_{i=0}^{U}\sum_{j=0}^{C}\sum_{p\in S'(i,j)}\min\{w_p, W(S')-k\}y_p$$

$$= \sum_{p\in S'}\min\{w_p, W(S')-k\}y_p$$

$$= (U+3)\sum_{p\in S'}\min\{w_p, W(S')-k\}x_p$$

$$\geq (U+3)(W(S')-k).$$

We use here the fact that $y_p = (U+3)x_p$ for $p \in S'$. □

We now show how to maintain the bounded cost property with $\beta = 10$. For this we need to use both the left- and right-hand sides of condition (7.31), and we use an argument similar to the one used in the proof of Lemma 7.8.

Lemma 7.11. Given any distribution μ over balanced sets that is consistent with y. If y changes to y' incurring a fractional cost of d, then the distribution μ can be modified to another distribution μ' over

balanced sets consistent with y' such that the total cost incurred is at most $10d$.

Proof. Suppose that y_p increases by ϵ and p lies in the class $S(i,j)$. Note that the balancing condition (7.31) holds for every size class different from i, and moreover for size class i the condition also holds for all cost classes higher than j. We apply the procedure used in Lemma 7.8 to size class i. Note that applying this procedure does not have any effect on size classes different from i, and we can thus iteratively balance cost classes starting from j down to 0 in size class i. To bound the cost, observe that the analysis in the proof of Lemma 7.8 only used the fact that the cost of the classes are geometrically decreasing. Thus, a similar analysis implies that the cost incurred is no more than $10\epsilon \cdot 2^j$. □

We conclude with the next theorem:

Theorem 7.12. There is an $O(\log^2 k)$-competitive algorithm for the caching problem in the general model.

7.4.3 The Fault Model

In this section, we study the fault model and show how to obtain an $O(\log k)$-competitive randomized caching algorithm for this model. Note that an $O(\log^2 k)$-competitive algorithm follows directly from the result for the general model. Recall that in the proofs for the bit model and the general model we crucially used the fact that the cost in the different classes is geometrically decreasing. However, this is not the case for the fault model, making the proof significantly more involved and requiring the use of a potential function so as to perform an amortized analysis.

We sort the n pages with respect to their size, i.e., $w_1 \leq w_2 \leq \cdots \leq w_n$. Let $x_1, \ldots, x_n$ be the LP solution at the current time step that satisfies the knapsack cover inequalities for all subsets. For each page p, let $y_p = \min\{1, 15 \cdot x_p\}$. Let S' denote the set of pages with $y_p < 1$. During the execution of the algorithm we maintain a grouping $\mathcal{G}$ of pages in S' into groups $G(i)$, $1 \leq i \leq \ell$. Each group $G(i)$ contains a

sequence of consecutive pages in S'. As the pages are ordered in non-decreasing order with respect to size, for any i the largest page size in group $G(i)$ is at most the smallest page size in $G(i+1)$.

Definition 7.3 (Good grouping). A grouping $\mathcal{G}$ of pages in S' is called *good* if it satisfies the following properties:

(1) For each i, $1 \leq i \leq \ell$, we have $\sum_{p \in S(i)} y_p \leq 12$.
(2) If $\sum_{p \in S'} y_p \geq 3$, then for each group i, $1 \leq i \leq \ell$, we have $\sum_{p \in G(i)} y_p \geq 3$. If $\sum_{p \in S'} y_p < 3$, then there is exactly one group $G(1)$ containing all the pages in S'.

We define $\sum_{p \in G(i)} y_p$ to be the *weight* of group $G(i)$.

Definition 7.4 (Balanced set). Given a good grouping G, a set of pages D is called balanced if the following two properties hold:

(1) If $y_p = 1$, then $A_p^D = 1$.
(2) For each i, the number of pages $|D \cap G(i)| = \sum_{p \in G(i)} A_p^D$ satisfies

$$\left\lfloor \sum_{p \in G(i)} y_p \right\rfloor \leq \sum_{p \in G(i)} A_p^D \leq \left\lceil \sum_{p \in G(i)} y_p \right\rceil . \tag{7.37}$$

The simulation procedure works as follows. At any time the algorithm maintains a good grouping $\mathcal{G}$ of the pages. It also maintains a probability distribution μ on balanced sets D which is consistent with y. At each step of the algorithm, as the value of y changes, the algorithm modifies the distribution μ to be consistent with y. Additionally, as y changes, the grouping $\mathcal{G}$ may also possibly change (so as to remain good), in which case a previously balanced set need not remain balanced anymore. In such a case, we also modify μ since only balanced sets can belong to the support of μ.

We first show that the size property holds for balanced sets D, and then show how to update $\mathcal{G}$ and μ as y changes, such that the cost property holds with $\beta = O(1)$ in an amortized sense.

Lemma 7.13. Let y be as defined above and let $\mathcal{G}$ be a good grouping with respect to y. Then any balanced set D with respect to $\mathcal{G}$ has size at least $W(S) - k$.

Proof. Let S' be the set of pages p for which $y_p < 1$. As D is balanced, each page with $y_p = 1$ belongs to D and hence it suffices to show that $\sum_{p \in S'} w_p A_p^D \geq W(S') - k$. If $W(S') - k \leq 0$, then we are already done. Henceforth, we assume that $W(S') - k > 0$.

The linear program constraint for the set S' implies that $\sum_{p \in S'} \min\{w_p, W(S') - k\} x_p \geq W(S') - k$. This implies that $\sum_{p \in S'} x_p \geq 1$ and so $\sum_{p \in S'} y_p \geq 15$. Hence, by the second condition for a good grouping, each group $G(i)$ has weight at least 3.

For each group $G(i)$ let $w_i(\min)$ and $w_i(\max)$ denote the smallest and largest page size in $G(i)$. Recall that for each i, we have that $w_i(\min) \leq w_i(\max) \leq w_{i+1}(\min)$. (Define $w_{\ell+1}(\min) = w_\ell(\max)$.) Let $m_i = \min(w_i(\min), W(S') - k)$ for $i = 1, \ldots, \ell + 1$. We lower bound the total size of pages in $D \cap S'$ as follows:

$$\begin{aligned}
\sum_{p \in S'} w_p A_p^D &\geq \sum_{p \in S'} \min\{w_p, W(S') - k\} A_p^D \\
&= \sum_{i=1}^{\ell} \sum_{p \in G(i)} \min\{w_p, W(S') - k\} A_p^D \\
&\geq \sum_{i=1}^{\ell} m_i \sum_{p \in G(i)} A_p^D \geq \sum_{i=1}^{\ell} m_i (-1 + \sum_{p \in G(i)} y_p) \\
&\geq \frac{2}{3} \sum_{i=1}^{\ell} m_i \sum_{p \in G(i)} y_p \qquad (7.38) \\
&= \frac{2}{3} \left(\sum_{i=1}^{\ell} m_{i+1} \sum_{p \in G(i)} y_p \right) - \frac{2}{3} \left(\sum_{i=1}^{\ell} (m_{i+1} - m_i) \sum_{p \in G(i)} y_p \right) \\
&\geq \frac{2}{3} \left(\sum_{i=1}^{\ell} m_{i+1} \sum_{p \in G(i)} y_p \right) - 8 \left(\sum_{i=1}^{\ell} (m_{i+1} - m_i) \right) \qquad (7.39)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{3}\left(\sum_{i=1}^{\ell} m_{i+1} \sum_{p \in G(i)} y_p\right) - 8m_{\ell+1} + 8m_1 \\
&\geq \frac{2}{3}\left(\sum_{i=1}^{\ell} \sum_{p \in G(i)} \min\{w_p, W(S') - k\} y_p\right) - 8(W(S') - k) \quad (7.40) \\
&\geq 2(W(S') - k).
\end{aligned}
$$

Here, inequality (7.38) follows since D is balanced, and hence for each $1 \leq i \leq \ell$,

$$\sum_{p \in G(i)} A_p^D \geq \left\lfloor \sum_{p \in G(i)} y_p \right\rfloor \geq -1 + \sum_{p \in G(i)} y_p,$$

and by observing that $\mathcal{G}$ is good and hence $\sum_{p \in G(i)} y_p \geq 3$ for each $1 \leq i \leq \ell$ and thus

$$-1 + \sum_{p \in G(i)} y_p \geq \frac{2}{3}\left(\sum_{p \in G(i)} y_p\right).$$

Inequality (7.39) follows since $m_{i+1} - m_i \geq 0$ for each $1 \leq i \leq \ell$, and since $\mathcal{G}$ is good, for each $1 \leq i \leq \ell$ we have that $\sum_{p \in G(i)} y_p \leq 12$. Finally, inequality (7.40) follows by considering the knapsack cover inequality for the set S' and observing that $y_p = 15x_p$ for each $p \in S'$:

$$
\begin{aligned}
&\sum_{i=1}^{\ell} \sum_{p \in G(i)} \min\{w_p, W(S) - k\} y_p \\
&\qquad = \sum_{p \in S'} \min\{w_p, W(S') - k\} 15 x_p \geq 15(W(S') - k). \qquad \square
\end{aligned}
$$

Lemma 7.14. As the solution y changes over time we can maintain a good grouping $\mathcal{G}$ and a consistent distribution on balanced sets with amortized cost at most a constant times the fractional cost.

Proof. The online fractional algorithm has the following dynamics. After a page p is requested, variable y_p can only increase (the page is

gradually evicted). This process stops when page p is requested again and y_p is set to zero. Whenever y_p changes, we need to modify the distribution μ on balanced sets D to remain consistent. Moreover, a change in y_p may change the structure of the groups. This happens if either the weight of $G(i)$ exceeds 12, or if it falls below 3, or if y_p becomes 1 and leaves the group $G(i)$ (recall that groups only contain pages q with $y_q < 1$). We view a change in y_p as a sequence of steps where y_p changes by an infinitesimally small amount ϵ. Thus, at each step exactly one of the following events happens.

Event 1: Variable $y_p < 1$ (of page p) increases or decreases by ϵ.
Event 2: The weight of group $G(i)$ reaches 12 units.
Event 3: The weight of group $G(i)$ drops to 3 units.
Event 4: The value of y_p for page p reaches 1 and p leaves the set $S(i)$.

We prove that in all cases the amortized cost of the online algorithm is at most $O(1)$ times the fractional cost. For amortization, we use the following potential function:

$$\Phi = 13 \sum_{p \in S'} y_p + 11 \sum_{i=1}^{\ell} \left| 6 - \sum_{p \in G(i)} y_p \right|.$$

In each possible event, let C_{on} be the total cost of the online algorithm. Let C_{f} be the fractional cost, and let $\Delta\Phi$ be the change in the potential function. We show that in each of the events:

$$\Delta C_{\text{on}} + \Delta\Phi \leq 405 \Delta C_{\text{f}}. \tag{7.41}$$

Since Φ is always positive, this will imply the desired result.

Event 1: Assume first that y_p such that $p \in G(i)$ is increased by ϵ. If y_p increases by ϵ it must be that x_p is increased by at least $\epsilon/15$. Thus, in the fault model the fractional cost is at least $\epsilon/15$.

To maintain consistency, we add p to ϵ measure of the sets D that do not contain p. However, this might make some of these sets unbalanced by violating (7.37). Suppose first that $s = \lceil \sum_{p \in G(i)} y_p \rceil$ does not change when y_p is increased by ϵ. In this case, we match the sets with $s + 1$

pages in $G(i)$ (the measure of these is at most ϵ) arbitrarily with sets containing $s-1$ pages, and transfer some page from the larger set (that does not lie in the smaller set) to the smaller set. An analogous argument works when s increases as y_p is increased. Note that after this step, the sets become balanced.

The total online cost is 3ϵ. Moreover, the potential change $\Delta\Phi$ is at most $13\epsilon + 11\epsilon = 24\epsilon$ and hence (7.41) holds. An analogous argument works if y_p is decreased (in fact it is even easier since the potential only decreases).

Event 2: Consider an event in which the total weight of a group $G(i)$ reaches 12 units. In this case, we split $G(i)$ into two sets such that their weight is as close to 6 as possible. Suppose one set is of size $6+x$ and the other is of size $6-x$ where $0 \le x \le 1/2$. Let $\Phi(s)$ and $\Phi(e)$ denote the potential function before and after the change, respectively. The contribution of the first term does not change. The second term corresponding to $G(i)$ initially is at least $11(12-6) = 66$ and the final contribution is $11(|6-(6-x)| + |6-(6+x)|) = 22x \le 11$. Thus, $\Delta\Phi = \Phi(e) - \Phi(s) = 11 - 66 \le -55$.

Next, we redistribute the pages in the original group $G(i)$ among the sets D such that they are balanced with respect to the two new groups. Observe that in the worst case, each set D might need to remove all the 12 pages it previously had and bring in at most $\lceil 6+x \rceil + \lceil 6-x \rceil \le 13$ new pages. Thus, the total cost incurred is at most 25. Again, (7.41) holds as the fractional cost C_{f} is 0 and the decrease in potential more than offsets the cost C_{on}.

Event 3: Consider the event when the weight of a group $G(i)$ decreases to three units. If $G(i)$ is the only group (i.e., $\ell = 1$) then all the properties of a good grouping still hold. Otherwise, we merge $G(i)$ with one of its neighbors (either $G(i-1)$ or $G(i+1)$). If $G(i)$ has a neighbor with weight at most 9, then we merge $G(i)$ with this neighbor. Note that before the merge each balanced set D has exactly three pages from $G(i)$ and hence it also remains balanced after the merge. Also, since $|6-3| + |6-x| \ge |6-(x+3)|$ for all $3 \le x \le 9$, the potential function does not increase in this case. Thus, (7.41) holds trivially.

Now suppose that all neighbors of $G(i)$ have weight greater than 9. Consider any such neighbor and let $x > 9$ be its weight. We merge $G(i)$ with this neighbor to obtain a group with weight $3 + x$ which lies in the range $(12, 15]$. Then, as in the handling of Event 2, we split this group into two groups with as close weight as possible. Since the weight is at most 15, the cost of balancing the sets D is at most $16 + 15 = 31$ (using an argument similar to that in Event 2). We now consider the change in potential. The only change is due to second terms corresponding to $G(i)$ and its neighbor (the first term does not matter since the total weight of pages in S' does not change upon merging or splitting). Before the merge, the contribution was $11 \cdot 3 + 11 \cdot (x - 6) = 11x - 33 \geq 66$. After the merge (and the split), the maximum value of the potential is obtained for the case when the size of the merged group is 15 which upon splitting leads to sets of size $7 + y$ and $8 - y$ where $y \leq 0.5$, in which case its value is $11(1 + y + 2 - y) = 33$. Thus, the potential function decreases by at least 33 while the online cost is at most 31, and hence (7.41) holds.

Event 4: Suppose some y_p increases to 1 and exits the group $G(i)$. Note that if $y_p = 1$, then all balanced sets D contain p. Thus, removing p from $G(i)$ keeps the sets balanced.

Let us first assume that the weight of $G(i)$ does not fall below 3 when p is removed. In this case, the groups and the balanced sets remain unchanged. Thus the online algorithm incurs zero cost. The first term of the potential decreases by 13, and the second term increases by at most 11, and hence (7.41) holds. Now consider the case when the weight of $G(i)$ falls below 3. We apply an argument similar to that for Event 3. If $G(i)$ can be merged with some neighbor without its weight exceeding 12, then we do so. This merge may cause some sets D to become unbalanced. However, this imbalance is no more than one page and can be fixed by transferring one page from each set to another, appropriately chosen, set. The total cost incurred in this case is at most 2. We now consider the change in potential. The first term decreases by 13. For the second term, the original group $G(i)$ contributes function $11(6 - (3 + x)) = 11(3 - x)$, with $x < 1$ and its neighbor contributes $11(|6 - z|)$ where $3 \leq z \leq 9$ is its weight. After the

merge, the second term corresponding to the merged group contributes $11(|6-(z+2+x)|)$ which is at most $11(|6-z|+(2+x))$. Overall, $\Delta\Phi \leq -13+11(2+x)-11(3-x) = 22x-24 < -2$. Thus (7.41) holds.

If we need to split the merged set, we note that the above analysis, showing that (7.41) holds, is also valid when $9 \leq z \leq 12$. Next, when this merged set is split, we can apply the analysis in Event 3, and then the potential function decreases by at least 33 units, while the cost incurred is at most 31, and hence (7.41) holds. □

We conclude with the next theorem:

Theorem 7.15. There is an $O(\log k)$-competitive algorithm for the caching problem in the fault model.

7.5 Notes

The results in this chapter are based on the work of Bansal et al. [14, 15]. The weighted caching problem was studied in [14], while the more general setting where pages have both sizes and fetching costs was studied in [15]. Converting a fractional view to an actual view has been considered previously by Bartal et al. [19] and Blum et al. [24]. Blum et al. [24] showed that for the unweighted caching problem it is possible to convert online a fractional view to an actual view, such that the expected cost incurred is at most twice the cost of the fractional view.

The unweighted paging problem is very well understood. In their seminal paper, Sleator and Tarjan [88] showed that any deterministic algorithm is at least k-competitive, and also showed that LRU (least recently used) is exactly k-competitive. They also considered the more general (h,k)-paging problem where the online algorithm with cache size k is compared against the offline algorithm with cache size h. They showed that any deterministic algorithm is at least $k/(k-h+1)$-competitive, and that LRU is exactly $k/(k-h+1)$-competitive. When randomization is allowed, Fiat et al. [48] designed the randomized marking algorithm which is $2H_k$-competitive against an oblivious

adversary. H_k denotes the kth harmonic number. They also showed that any randomized algorithm is at least H_k-competitive. Subsequently, McGeoch and Sleator [81] gave a matching H_k-competitive algorithm, and Achlioptas et al. [1] gave another H_k-competitive algorithm which is easier to state and analyze. For (h,k)-paging, Young [91] gave a $2\ln(k/(k-h))$-competitive algorithm (ignoring lower order terms) and showed that any algorithm is at least $\ln(k/(k-h))$-competitive. There has been extensive work on paging along several other directions, and we refer the reader to the excellent book by Borodin and El-Yaniv [28] for further details.

For weighted paging, a (tight) k-competitive deterministic algorithm follows from the more general work of Chrobak et al. [40] on the k-server problem in trees (see below). Subsequently, Young [92] gave a tight $k/(k-h+1)$-competitive deterministic algorithm for the more general (h,k)-paging problem. The status of the randomized competitiveness of the weighted paging problem remained open until it was fully settled by Bansal et al. [14]. Irani [66] gave an $O(\log k)$-competitive algorithm for the two weight case, i.e. when each page weight is either 1 or some fixed $M > 1$. Blum et al. [25] gave an $O(\log^2 k)$-competitive algorithm for the case of $n = k + 1$ pages. Later, Fiat and Mendel [50] gave an improved $O(\log k)$ competitive algorithm for the case of $n = k + c$ pages, where c is a constant. For large n, however, no $o(k)$-competitive algorithm was known even for the case of three distinct weights.

Paging can be viewed as a special case of the much more general and challenging k-server problem. Suppose there are k-servers located in an n-point metric space. The requests are given at the points of the metric and they are served by moving a server to the requested point. The goal is to minimize the overall distance traveled by the servers. The unweighted paging problem is exactly the k-server problem on a uniform metric space. The weighted paging problem is equivalent (up to an additive constant) to the k-server problem on a star metric in which the distance between any two pages a and b is $(w(a) + w(b))/2$, where $w(\cdot)$ denotes the page weights.

The k-server problem has a fascinating history and substantial progress has been made on deterministic algorithms for the problem.

It is known that any deterministic algorithm must be at least k-competitive on any metric space with more than k points. Fiat et al. [51] gave the first algorithm for which the competitive ratio was only a function of k. Their algorithm was $O((k!)^3)$-competitive. After a series of results, a breakthrough was achieved by Koutsoupias and Papadimitriou [76] who gave an almost tight $2k-1$ competitive algorithm. This is still the best known bound (for both deterministic and randomized algorithms) for general metric spaces. A tight competitive factor of k is known for certain special cases, e.g., for trees Chrobak et al. [40]. We refer the reader to [28] for more details on the k-server problem.

Nevertheless, randomized algorithms for the k-server problem remain poorly understood. No lower bound better than $\ln k$ is known for any metric space. Moreover, from the work of Bartal et al. [18] and Bartal et al. [20], it follows that no metric space with more than k points can admit an $o(\log k/\log\log k)$-competitive algorithm. A widely believed conjecture is that $O(\log k)$-competitive algorithms exist for every metric space. In a breakthrough result, Bartal et al. [19] gave a polylog(N) competitive algorithm for the metrical task system problem (see definitions in Section 6) that implies a polylog(k)-competitive algorithm for the k-server on a space with $k+c$ points, where c is a constant independent of k. This guarantee was improved by Fiat and Mendel [50] to $O(\log^2 k \log\log k)$. However, for n much larger than k, no algorithms with sublinear competitive ratio are known except for very few special cases. Besides paging and weighted paging with two weights, a polylogarithmic competitive algorithm is known for a special subclass of certain well-separated spaces [87]. Csaba and Lodha [44] gave an $O(n^{2/3})$ competitive algorithm, which is $o(k)$ competitive for $n = o(k^{3/2})$, for n uniformly spaced points on a line.[2]

Generalized caching where the page sizes are also non-uniform is substantially harder. In contrast to uniform page size caching, even the offline version of the problem is NP-hard, as it captures the knapsack

[2] A generalization of this result was considered by Bartal and Mendel [21], who proposed a $\Delta^{1-\epsilon}\text{polylog}\, k$ competitive algorithm for bounded growth metrics with diameter Δ. Unfortunately, their result seems to have a serious error [M. Mendel, personal communication].

problem as a special case.[3] Following a sequence of results [2, 43, 65], Bar-Noy et al. [16] gave a 4-approximation for the offline problem based on the local-ratio technique. This is currently the best known approximation for (offline) generalized caching. For the online case, it is known that LRU is $(k+1)$-competitive for the bit model and also for the fault model [65], where k denotes the ratio between cache size and the size of the smallest page. Later on, Cao and Irani [35] and Young [94] gave a $(k+1)$-competitive algorithm for the general model based on a generalization of the greedy-dual algorithm of Young [92]. An alternate proof of this result was obtained by Cohen and Kaplan [43]. When randomization is allowed, Irani [65] designed an $O(\log^2 k)$-competitive algorithm for both fault and bit models. These algorithms are very complicated and are based on an approach combining offline algorithms with the randomized marking algorithm. For the general model, no $o(k)$ randomized algorithms are known. There has been extensive work on caching in other directions, and we refer the reader for further details to the excellent book by Borodin and El-Yaniv [28] and to the survey by Irani [64] on paging.

[3] It remains NP-hard for the bit model. For the fault model, it is open whether the problem is polynomially solvable [65].

8

Load Balancing on Unrelated Machines

In this section, we show how to use the online primal–dual approach to design an optimal online algorithm for the problem of load balancing on unrelated machines. In this setting, there is a set of m machines $\mathbb{S} = \{s_1, s_2, \ldots, s_m\}$ and a set of jobs $\mathbb{R}$. There is a load $p(i, j)$ associated with each job $r_i \in \mathbb{R}$ and machine $s_j \in \mathbb{S}$, corresponding to the processing time of r_i on s_j. The load on each machine is defined to be the sum of the processing times of the jobs that are assigned to the machine. Our goal is to distribute the jobs between the m machines so as to minimize the maximum load on a machine. In the online setting, the jobs arrive one-by-one and need to be assigned to a machine upon arrival. The assignment of a job to a machine is final and it cannot be changed at a later point of time.

8.1 LP Formulation and Algorithm

The first idea we need is that of "guessing" the value of the optimum. That is, our online algorithm is going to guess the value of the maximum load Λ^* on a machine in an optimal assignment of the jobs. This will

be done by starting from value $\alpha = \min_{j=1}^{m}\{p(1,j)\}$ and *doubling* the guess whenever needed, until $\alpha \geq \Lambda^*$. We design an algorithm that never assigns more than $\alpha \cdot O(\log m)$ units of load to any machine. The algorithm guarantees success in assigning all jobs when it is given a value $\alpha \geq \Lambda^*$. When the algorithm is given a value $\alpha < \Lambda^*$ it may fail. Each time the value of α is doubled we "forget" about all previous assignments and run the online algorithm on the remaining jobs. Since the assignments we forget about are bounded by a geometric sequence, it is not hard to see that doubling can deteriorate the competitive ratio by at most a multiplicative factor of 4.

Given a guess α, we define the normalized load of job r_i on machine s_j to be $\tilde{p}(i,j) = p(i,j)/\alpha$. Upon arrival of job j arrives we are going to consider only machines for which $\tilde{p}(i,j) \leq 1$. Assuming $\alpha \geq \Lambda^*$, the job can only be processed by the optimal solution on such machines. If there is no such machine then our guess of Λ^* is certainly wrong and the algorithm fails. Next, we formulate the problem as a packing linear program. For each job r_i, let $\mathbb{S}(r_i)$ be the set of machines for which $\tilde{p}(i,j) \leq 1$. We have a variable $y(i,j)$ indicating that job r_i is assigned to machine s_j. The objective function is to maximize the number of jobs assigned to the machines. The formulation appears as the dual program (maximization) in Figure 8.1 along with its corresponding primal program.

Let $N = |\mathbb{R}|$ be the number of jobs. The important observation is that when we guess a value $\alpha \geq \Lambda^*$, then it is possible to assign all the jobs to the machines without exceeding the load. This means that the value of the optimal dual solution is exactly N and there is no primal solution that has value strictly less than N. We are now ready to state the algorithm:

Primal	Dual
Minimize: $\sum_{s_j \in \mathbb{S}} x(j) + \sum_{r_i \in \mathbb{R}} z(i)$	Maximize: $\sum_{r_i \in \mathbb{R}} \sum_{s_j \in \mathbb{S}(r_i)} y(i,j)$
subject to:	subject to:
$\forall r_i \in \mathbb{R}, s_j \in \mathbb{S}(r_i)$: $\tilde{p}(i,j)x(j) + z(i) \geq 1$	$\forall r_i \in \mathbb{R}$: $\sum_{s_j \in \mathbb{S}(r_i)} y(i,j) \leq 1$
	$\forall s_j \in \mathbb{S}$: $\sum_{r_i \in \mathbb{R}, s_j \in \mathbb{S}(r_i)} \tilde{p}(i,j)y(i,j) \leq 1$

Fig. 8.1 A primal–dual pair for the load balancing problem on unrelated machines.

Load balancing algorithm:

Initially: $\forall j,\ x(j) \leftarrow 1/(2m)$.
When a new job r_i arrives:

(1) If there is no machine s_j such that $\tilde{p}(i,j) \leq 1$, or if there exists a machine s_j with $x(j) > 1$, return "failure".
(2) Otherwise:
   (a) Let $s_\ell \in \mathbb{S}(r_i)$ be a machine minimizing $\tilde{p}(i,\ell)x(\ell)$.
   (b) Assign request r_i to machine s_ℓ
   (c) Set $z(i) \leftarrow 1 - \tilde{p}(i,\ell)x(\ell)$ and $y(i,\ell) \leftarrow 1$.
   (d) $x(\ell) \leftarrow x(\ell)(1 + \frac{\tilde{p}(i,\ell)}{2})$.

Theorem 8.1. If there is a feasible dual solution that assigns all jobs, then the algorithm assigns all jobs with normalized load $O(\log m)$.

Proof. To prove the theorem we prove the following claims:

(1) The load on each machine is at most $O(\log m)$.
(2) If the algorithm fails in line (1), then there is a feasible primal solution whose value is strictly smaller than $|\mathbb{R}| = N$.

Proof of (1): Note that the algorithm never assigns a job to a machine s_j with $x(j) > 1$ (otherwise, it already fails in line (1)). Also, the initial value of $x(j)$ is $1/2m$ and $x(j) \leq 3/2$ always holds, since $\tilde{p}(i,j) \leq 1$. Let $\mathbb{R}(s_j)$ be the set of jobs that the algorithm assigned to machine s_j. We thus have the following inequalities:

$$\frac{3}{2} \geq x(j) \geq \frac{1}{2m} \prod_{r_i \in \mathbb{R}(s_j)} \left(1 + \frac{\tilde{p}(i,j)}{2}\right) \geq \frac{1}{2m} \prod_{r_i \in \mathbb{R}(s_j)} \left(\frac{3}{2}\right)^{\tilde{p}(i,j)}$$
$$= \frac{1}{2m} \exp\left(\ln\left(\frac{3}{2}\right) \sum_{r_i \in \mathbb{R}(s_j)} \tilde{p}(i,j)\right).$$

Simplifying, we get that:

$$\sum_{r_i \in \mathbb{R}(s_j)} \tilde{p}(i,j) \;\leq\; \frac{\ln(3m)}{\ln\left(\frac{3}{2}\right)} = O(\log m).$$

Note that this bound also holds in case of failure.

Proof of (2): First note that the algorithm produces a feasible primal solution. This is true since we set, for each job r_i, $z(i) \leftarrow 1 - \tilde{p}(i,\ell)x(\ell)$, where s_ℓ is the machine minimizing the value $\tilde{p}(i,\ell)x(\ell)$. Thus, we satisfy all the new primal constraints that have arrived in the iteration of job r_i. Since the variables $x(j)$ are monotonically non-decreasing, once a constraint is satisfied in an iteration, it remains so in subsequent iterations. Next, observe that whenever we assign job r_i to machine s_j the change in the value of the primal objective function is

$$1 - \tilde{p}(i,\ell)x(\ell) + \frac{\tilde{p}(i,\ell)x(\ell)}{2} = 1 - \frac{\tilde{p}(i,\ell)x(\ell)}{2}.$$

Note, however, that the change in $x(\ell)$ due to the assignment of job r_i is exactly $\tilde{p}(i,\ell)x(\ell)/2$. Let $x(j)_{\text{init}}$ be the initial value of $x(j)$ (which is $1/2m$). Thus, by this observation, at any time during the execution of the algorithm the value of the primal objective function is

$$\begin{aligned} P &= \sum_{j=1}^{m} x(j)_{\text{init}} + N - \sum_{j=1}^{m} (x(j) - x(j)_{\text{init}}) \\ &= 2\sum_{j=1}^{m} x(j)_{\text{init}} + N - \sum_{j=1}^{m} x(j) = 1 + N - \sum_{j=1}^{m} x(j), \end{aligned}$$

where N is the number of jobs. Assume now that there exists some variable $x(j) > 1$. This means that we have a primal solution with value strictly less than N, meaning that there can be no dual solution with value exactly N. We now have a certificate that $\alpha < \Lambda^*$, concluding the proof. □

8.2 Notes

The results in this chapter are based on the work of Buchbinder and Naor [34]. The algorithm described in this section and its analysis are actually a primal–dual view of a previous algorithm by Aspnes et al. [8].

Many load balancing models were studied in the literature. Perhaps the simplest one is the *identical machines* model. There are m identical machines and the input is n jobs arriving online, each having a machine-independent load. An assignment of a job is final and cannot be changed. For this model, Graham [61] proved that the simple greedy heuristic that assigns a job to the least loaded machine is $2 - 1/n$-competitive. Another popular model is the *restricted assignment* model. In this model, each job can be assigned only to a prespecified subset of the machines (and not to all of them, as in the identical machines model). For this model, [13] analyzed the performance of the same greedy strategy, proving that it is $\Theta(\log m)$-competitive. More refined performance measures of the greedy strategy were later studied in [34, 58]. For further discussion of online load balancing in a variety of models we refer the reader to [12].

9

Routing

In this section, we study network routing problems. We have already discussed two simple routing algorithms and a primal–dual approach for solving routing problems in Section 4.4.2. In this section, we will design more complex routing algorithms, taking into account other objective functions.

Consider a network modeled by a graph $G = (V, E)$ ($|V| = n$, $|E| = m$), which can either be directed or undirected. The edges in the graph have capacities, denoted by $u : E \to \mathbb{N}$, providing an upper bound on the sum of the flows of the routes that can be packed into an edge. The set of routing requests is $\mathbb{R}$ and, for simplicity, each request $r_i \in \mathbb{R}$ is associated with a bandwidth demand of one unit[1] between a source vertex s_i and a target vertex t_i. In order to serve a request r_i one should allocate bandwidth for the request on paths that connect the source vertex s_i to the target vertex t_i. There are several common ways by which this can be done. The setting in which each request has to be served via a single path is referred to as *unsplittable routing*. A less restrictive setting in which each request can be served via multiple

[1] The results in this section can be extended, with obvious limitations, to handle scenarios in which requests have different bandwidth demands.

routes is called *splittable routing*. We associate each request with a set of allowed paths (routes) $\mathbb{P}(r_i)$, capturing the *fixed routes* model, in which requests can only be served via a *unique* given path, as a special case. Let $b(r_i)$ be the sum of all bandwidth allocations assigned to request r_i on all paths $P \in \mathbb{P}(r_i)$. The *total bandwidth* of a routing solution is the total bandwidth allocated to all the requests. A *feasible* routing solution is an allocation of bandwidth to requests that does not violate any of the edge capacities. When the routing solution is infeasible, the *load* on an edge is the total bandwidth allocated to it divided by its capacity. The load of a routing solution is the maximum load taken over all edges.

An important parameter that is used in our analysis is U, which is defined to be the minimum value by which the capacities in the network need to be multiplied so as to obtain a feasible splittable solution that routes all requests. When routes are fixed, U reduces to the maximum, taken over all edges, of the number of routes that pass through an edge, divided by its capacity.

The issue of whether requests have to be fully served or not distinguishes between different routing models. *All-or-nothing* routing means that a request has to be allocated a total bandwidth (splittable or unsplittable) of one unit. Other models relax this requirement and allow the routing algorithm to allocate requests less than one unit of bandwidth.

Routing algorithms are designed to achieve several natural goals. One goal is to maximize the *utility* of the network which is the total bandwidth allocated to all requests. In a somewhat dual setting, the routing algorithm is not allowed to reject any of the requests, in which case the goal is to minimize the maximum load. Another important routing goal is *fairness*. Fairness can be defined in more than one way, however, a common notion of fairness is called *max–min fairness*. To define a fair routing solution, we consider the bandwidth allocation to the requests ($b(r_i)$ to request r_i) as a vector in which the entries (allocations) are sorted from small to large. This vector is called a *bandwidth vector*. A max–min fair routing solution is then an allocation of bandwidth to requests defining a lexicographically maximal bandwidth vector. An intuitive way of viewing a max–min fair solution is that

the bandwidth allocation to a request r_i cannot be increased without decreasing the bandwidth allocated to requests that have received no more than the bandwidth allocated to r_i.

An even more general fairness measure studied in the literature is the notion of coordinate-wise competitive solution. A routing solution is called γ_c-*coordinate-wise competitive*, if for every i, the ith coordinate of the bandwidth vector is at least $1/\gamma_c$ times the ith coordinate in any feasible routing solution. The beauty of this definition is that a γ_c-coordinate-wise competitive routing approximates *all* possible routings. In particular, it approximates the max–min fair routing, as well as the routing solution that maximizes the total bandwidth allocated, achieving in a sense a solution which is the "best of all worlds."

Two parameters are of particular interest in routing problems. The first one is the amount of *bandwidth* that the algorithm routes with respect to an optimal routing, and the second one is the maximum *load* on the edges. A (c_1, c_2)-competitive routing algorithm routes at least $1/c_1$ of the maximum possible bandwidth, while guaranteeing that the load on each edge is at most c_2. With this notation in mind we re-examine the first algorithm in Section 4.4.2 and conclude that it is $(3, O(\log n))$-competitive. This algorithm is actually a bicriteria competitive algorithm that routes a constant fraction of the optimal number of requests while incurring a load of $O(\log n)$.

It turns out that getting a uni-criteria competitive algorithm (i.e., an $(1, O(\log n))$-competitive algorithm) is a crucial non-trivial step for getting better routing solutions for many routing goals.[2] In particular, we will show that given such an algorithm it is easy to design an algorithm that achieves fair routing. A simple example shows that such a result is optimal for an online algorithm. In addition, the generic algorithm we design here generates an unsplittable all-or-nothing routing; however, to allow the use of the algorithm in a wide variety of routing models, its performance is compared to a splittable optimal routing which is allowed to allocate to each request (total) bandwidth of at

[2] Note that we can easily transform a (c_1, c_2)-competitive algorithm to a $(c_1 \cdot c_2, 1)$-competitive algorithm by scaling down all allocated bandwidth. However, obtaining a $(1, c_1 \cdot c_2)$-competitive factor is not as easy, since requests can only be allocated bandwidth of at most 1.

most 1. It turns out that this stronger performance allows the use of the generic algorithm as a "building block" for the design of online routing solutions for several models and objectives, yielding improved bounds. Here we will only show one such application for getting a fair routing.

9.1 A Generic Routing Algorithm

In this section, we design a generic online routing algorithm which is based on the primal–dual approach. The algorithm generates in an online fashion an unsplittable all-or-nothing routing which is $(1, O(\log n))$-competitive with respect to all splittable routings. To this end, maintaining a single primal solution is not sufficient, leading us to the simultaneous maintainance of several primal solutions that will be used throughout for making clever routing decisions. We will use the same primal–dual pair that was used in Section 4.4.2, see Figure 9.1 (same as Figure 4.2).

The algorithm decomposes the graph $G = (V, E)$ into graphs $G_0, G_1, \ldots, G_k$. For each j, the vertex set of G_j is V. The edges of G_j are all the edges of G having capacity at least m^j. The capacity of each edge in the jth copy, G_j, is set to $u(e, j) \leftarrow \min\{u(e), m^{j+2}\}$. Let G_k be the last copy of the graph which is non-empty (i.e., the maximum capacity in G, $u(\max) \leq m^k$). The algorithm maintains a primal solution in each copy of the graph. We denote by $x(e, j)$ and $z(r_i, j)$ the primal variables corresponding to the jth copy. Let $u(\min, j)$ be the minimal edge capacity in the jth copy (which is at least m^j).

Primal		Dual	
Minimize:	$\sum_{e \in E} u(e)x(e) + \sum_{r_i} z(r_i)$	Maximize:	$\sum_{r_i} \sum_{P \in \mathbb{P}(r_i)} f(r_i, P)$
subject to:		subject to:	
$\forall r_i \in \mathbb{R}, P \in \mathbb{P}(r_i)$:	$\sum_{e \in P} x(e) + z(r_i) \geq 1$	$\forall r_i \in \mathbb{R}$:	$\sum_{P \in \mathbb{P}(r_i)} f(r_i, P) \leq 1$
		$\forall e \in E$:	$\sum_{r_i \in \mathbb{R}, P \in \mathbb{P}(r_i) \mid e \in P} f(r_i, P) \leq u(e)$
	$\forall r_i, z(r_i) \geq 0, \forall e, x(e) \geq 0$	$\forall r_i, P$:	$f(r_i, P) \geq 0$

Fig. 9.1 The splittable routing problem (maximization) and its corresponding primal problem.

Routing algorithm:

Initially, $\forall j$: $x(e,j) \leftarrow u(\min,j)/m \cdot u(e,j)$.
When new request $r_i = (s_i, t_i, \mathbb{P}(r_i))$ arrives:

(1) Consider all copies of G from G_k to G_0. In each copy G_j:
  (a) Let $P(r_i,j) \in \mathbb{P}(r_i,j)$ be the shortest path with respect to $x(e,j)$ and let α be the length of $P(r_i,j)$.
  (b) If $\alpha < 1$:
    (i) Route the request on $P(r_i,j)$.
    (ii) For each edge e in $P(r_i,j)$: $x(e,j) \leftarrow x(e,j)(1 + 1/u(e,j))$.
    (iii) $z(r_i,j) \leftarrow 1 - \alpha$.
  (c) Else ($\alpha > 1$):
    (i) If the total bandwidth routed in this step in G_j is less than $u(\min,j)$, and the current request can be routed in G_j, route the request in an arbitrary feasible path $P \in \mathbb{P}(r_i,j)$.
  (d) If the request is routed — finish.
(2) Reject requests that got rejected from all copies.

The following claims are used for analyzing the algorithm.

Lemma 9.1. Let N_j be the total number of requests that are introduced to the jth copy. Let M be the maximum total bandwidth of any feasible splittable routing in G_j (out of N_j). Then, the algorithm accepts at least M requests in G_j, and the load on each edge in G_j is $O(\log n)$.

Proof. First, observe that when the algorithm decides to route a request in step (1b), the total primal value maintained in the jth copy increases by $(1-\alpha) + \sum_{e \in P(r_i,j)} x(e,j) = 1$. When a request is rejected from

the jth copy, the primal value in the jth copy does not change. Second, observe that the primal solution maintained in each copy is feasible with respect to the requests introduced to this copy. This follows, since, if the shortest path in $\mathbb{P}(r_i, j)$ is already at least 1, the constraints of the new request are all satisfied. If the shortest path is of length $\alpha < 1$, then the algorithm updates $z(r_i, j)$ to be $1 - \alpha$ to make the current new set of constraints feasible. All previous constraints remain feasible. Finally, note that initially the total primal value in the jth copy is

$$\sum_{e \in E} u(e,j) \frac{u(\min, j)}{m \cdot u(e,j)} = u(\min, j).$$

Assume to the contrary that the algorithm routes in step (1b) a total bandwidth $M' < M - u(\min, j)$. This immediately implies that we have a feasible primal solution of value strictly less than $M - u(\min, j) + u(\min, j) = M$, contradicting the fact that we have a feasible dual solution of value M (out of N_j). This means that $M - M'$ is at most $u(\min, j)$ and thus, at least $M - M'$ (or zero, if this value is negative) are routed in G_j in step (1c), proving the first part of the claim.

We next prove the second part of the claim. The initial value of each $x(e,j)$ is $u(\min, j)/m \cdot u(e,j)$. Each time a new request is routed on edge e in step (1b), $x(e,j)$ is multiplied by $1 + 1/u(e,j) \leq 2$. The algorithm never routes requests in step (1b) on edges with $x(e,j) > 1$, thus, $x(e,j) \leq 2$. Let $\mathbb{R}(e)$ be the set of requests that are routed on an edge e. We get that:

$$\begin{aligned}
\frac{u(\min, j)}{m \cdot u(e,j)} \exp\left(\frac{\ln 2}{u(e,j)} \sum_{r_i \in \mathbb{R}(e)} 1\right) &= \frac{u(\min)}{m \cdot u(e,j)} \prod_{r_i \in \mathbb{R}(e)} 2^{1/u(e,j)} \\
&\leq \frac{u(\min, j)}{m \cdot u(e,j)} \prod_{r_i \in \mathbb{R}(e)} \left(1 + \frac{1}{u(e,j)}\right) \\
&\leq 2.
\end{aligned}$$

Simplifying, the total bandwidth that the algorithm routes on edge e in step (c) is $u(e,j)O(\log(m \cdot u(e,j)/u(\min, j)))$. Since $u(e,j) \leq m^2 u(\min, j)$, the latter expression is equal to $u(e,j)O(\log n)$. The total

bandwidth that the algorithm routes on edge e in the jth copy in step (d) is at most $u(\min, j)$, thus completing the proof. □

Theorem 9.2. The routing algorithm is $(1, O(\log n))$-competitive with respect to all splittable routing solutions.

Proof. Let M be the maximum bandwidth that can be routed splittably in G (without violating the constraints). The solution that routes bandwidth of M can be decomposed into routing paths $P_1^*, P_2^*, \ldots, P_\ell^*$, and let $b_1^*, b_2^*, \ldots, b_\ell^*$ be the bandwidth allocated to each of the paths. Note that each request can be served via multiple routes and get a total bandwidth in the interval $[0, 1]$.

We start by proving that the total bandwidth that the algorithm routes is at least M. For each of the ℓ routing paths, let $w_j \leftarrow \min_{e \in P_j^*} u(e)$, i.e., w_j is the minimum capacity used in path P_j^*. We partition the paths into separate groups $M_0, M_1, \ldots, M_k$. Group M_i consists of all paths for which $m^i \leq w_j \leq m^{i+1}$. Let $|M_i|$ be the total bandwidth of all paths in M_i. Note that $\sum_{i=1}^{k} |M_i| = M$. We prove by backward induction that the total bandwidth allocated by the algorithm in levels G_j to G_k is at least $|M_j| + |M_{j+1}| + \cdots + |M_k|$. Therefore, the total bandwidth that the algorithm allocates in levels G_0 to G_k is at least M.

Induction basis: For $j = k$, the graph G_k consists of edges with capacity at least m^k. In this graph, the capacities of the edges are the same as their capacities in the graph G. Group M_k consists of paths in which the minimum capacity is at least m^k. Thus, all the paths in M_k also exist in the graph G_k. By Lemma 9.1, the algorithm routes a total bandwidth of at least $|M_k|$ out of N (all the requests).

Inductive step: Let G_j be any level $j < k$. Let $M'_{j+1}, M'_{j+2}, \ldots, M'_k$ be the groups of requests that were routed by the algorithm in $G_{j+1}, \ldots, G_k$. By the inductive hypothesis, $|M'_{j+1}| + |M'_{j+2}| + \cdots + |M'_k| \geq |M_{j+1}| + |M_{j+2}| + \cdots + |M_k|$.

We consider the set of paths S in $(M_j \cup M_{j+1} \cup \cdots \cup M_k)$ that do not belong to requests that are routed in $G_k, G_{k-1}, \ldots, G_{j+1}$. Let $|S|$ be

the total bandwidth allocated by the feasible solution on these paths. These paths all belong to requests that are given to the graph G_j (i.e., counted as part of N_j). We claim that it is possible to route in G_j a total bandwidth of at least $\min\{|M_j|, |S|\}$ out of the requests that are given to the algorithm in level j. In order to prove this, we prove that if we take any part of the total flow of the paths in S with total bandwidth of at most $|M_j|$, it is possible to route this flow on the graph G_j without violating capacities.

Set S consists of paths in groups of at least M_j, thus each path $P_j^* \in S$ contains only edges with capacity at least m^j. G_j contains all the edges whose capacity is at least m^j, so the path P_j^* exists in the graph G_j. Each path in M_j also contains an edge with capacity at most m^{j+1}. This means that the total bandwidth allocated by the feasible solution on all paths that belong to M_j is at most m^{j+2}. The capacity of the edges in G_j is restricted to $\min\{u(e), m^{j+2}\}$. Thus, if we take part of the total flow of the paths in S with a total bandwidth of at most $|M_j| \leq m^{j+2}$, it is possible to route this flow on the graph G_j without violating the capacities.

By the above claim and Lemma 9.1, the total bandwidth of requests that are routed in G_j is at least $\min\{|M_j|, |S|\}$. If $|S| \geq |M_j|$, then a total bandwidth of at least $|M_j|$ will be routed in G_j, and since $|M'_{j+1}| + |M'_{j+2}| + \cdots + |M'_k| \geq |M_{j+1}| + |M_{j+2}| + \cdots + |M_k|$ the induction hypothesis holds. If $|S| < |M_j|$, then bandwidth of at least $|S| \geq (|M_j| + |M_{j+1}| + \cdots + |M_k|) - (|M'_{j+1}| + |M'_{j+2}| + \cdots + |M'_k|)$ will be routed in G_j, and the induction hypothesis holds again.

To prove the second part of the theorem consider an edge e with capacity $m^j \leq u(e) < m^{j+1}$. Its capacity in levels $\ell > j$ is zero. Therefore, no requests are routed through e in G_ℓ for $\ell > j$. The capacity of e in levels $j, j-1, j-2$ is $u(e)$ and in levels $j-3, j-2, \ldots, 0$ the capacity drops to $m^{j-1}, m^{j-2}, \ldots, m^2$. Thus, the total capacity of the copies of edge e in all levels is at most four times its capacity. In each level G_j, the ratio between the maximum and minimum capacity of the edges is at most m^2. Thus, the total bandwidth of the requests that are routed on edge e in each level is at most $u(e, j)O(\log n)$. Therefore, the total number of requests routed on edge e in all levels is at most

$O(\log n)$ times the sum of the capacities of e in all levels, which remains $O(\log n)$ times the capacity of edge e in G. □

9.2 Achieving Coordinate-Wise Competitive Allocation

We now show how the generic routing algorithm can be used for achieving a fair allocation. We design an almost optimal online algorithm for achieving a coordinate-wise routing solution. In this setting, the algorithm should output an unsplittable routing and assign bandwidth $b \in [0,1]$ to each request. We design an $O(1/\epsilon \log n \log U (\log \log U)^{1+\epsilon})$-competitive algorithm for any $\epsilon > 0$ and prove an almost matching lower bound of $\Omega(\log n \log U + \log U \log \log U)$ even when splittable routing is allowed. The algorithm is quite simple. It considers copies of the graph referred to as levels. In levels $\ell = 0, 1, 2, \ldots$ we multiply all edge capacities by 2^ℓ.

Algorithm: When a request $r_i = (s_i, t_i, \mathbb{P}(r_i))$ arrives:

(1) Run the generic routing algorithm on levels $\ell = 0, 1, 2, \ldots$
(2) Route the request in the lowest level ℓ in which the algorithm accepts the request.
(3) Assign the request bandwidth of $(\epsilon/(c \log n 2^\ell (1+\ell)(H(1+\ell))^{1+\epsilon}))$ where c is a constant and $H(\cdot)$ is the harmonic number.

Theorem 9.3. The algorithm is $O(1/(\epsilon \log n \log U (\log \log U)^{1+\epsilon}))$-coordinate-wise competitive.

We first prove a useful lemma that will help us in the analysis:

Lemma 9.4. If it is possible to route M (out of N) requests in a splittable way when multiplying the capacities of the edges in G by 2^k, then the following holds:

- The algorithm routes at least M requests using levels 1 to k.
- The total number of requests that are routed on edge e in level k is at most $2^k u(e) O(\log n)$.

Proof. Let M' be the group of requests that are routed in levels 1 to $k-1$. If $|M'| \geq |M|$, then we are done. Otherwise, it is possible to route in level k a total bandwidth of at least $|M| - |M'|$ out of the requests that are rejected by levels 1 to $k-1$. Thus, by Theorem 9.2, the algorithm routes at least $|M| - |M'|$ requests in level k, and we are done. The second claim is immediate from Theorem 9.2. □

Proof [of Theorem 9.3]. To prove that the algorithm is γ_c-coordinate-wise competitive, we need to show that the value of any coordinate in the bandwidth vector of the solution we generate is at least $1/\gamma_c$ of this coordinate in any other feasible solution. Consider the ith coordinate of the bandwidth vector, i.e., the ith "poorest" request. We need to show that if there exists a solution that assigns the ith coordinate bandwidth b, then our algorithm assigns to at least $N - i + 1$ requests bandwidth $b \cdot \Omega(\epsilon/(\log m \log U(\log\log U)^{1+\epsilon}))$. Now, if there exists a feasible solution that assigns bandwidth b to the ith coordinate, then it must assign bandwidth at least b to at least $N - i + 1$ coordinates. This means that there exists a feasible solution that routes at least $N - i + 1$ requests when multiplying the capacities in the graph G by $1/b$. Thus, by Lemma 9.4, at least $N - i + 1$ requests will be routed in the first k levels where $2^k \leq 2/b$. All these requests are, therefore, assigned bandwidth at least

$$\frac{b \cdot \epsilon}{2c \log m(1+k)(H(1+k))^{1+\epsilon}} = b \cdot \Omega\left(\frac{\epsilon}{\log m \log U(\log\log U)^{1+\epsilon}}\right).$$

The last equality follows since, by Lemma 9.4 and the definition of U, all the requests will be routed until level k, such that $2^k \leq 2U$.

It is left to prove that the algorithm does not violate the capacity of any of the edges. By Lemma 9.4, the number of requests routed on an edge e in level k is at most $2^k u(e) O(\log n)$. Thus, when we choose a large enough constant c, the total bandwidth assigned to each edge e is at most:

$$\sum_{j=0}^{\infty} 2^j \cdot u(e) O(\log n) \frac{\epsilon}{c \log n 2^j (1+j)(H(1+j))^{1+\epsilon}}$$
$$\leq \frac{u(e)}{c'} \sum_{j=1}^{\infty} \frac{\epsilon}{j(H(j))^{1+\epsilon}} \leq u(e). \qquad \square$$

Lower bounds: In [59], a lower bound of (approximately) $\Omega(\log n + \log U \log\log U)$ was proved. We improve on this lower bound obtaining an almost matching lower bound for the problem.

Lemma 9.5. Any deterministic algorithm (splittable or unsplittable) is $\Omega(\log n \log U)$-coordinate-wise competitive.

Proof. Let $G = (V, E)$ be a directed line with n nodes. Assume without loss of generality that n is a power of 2. The first U requests are going to be routed either from node 1 towards node $n/2$, or from node $n/2 + 1$ towards node n. To generate such requests, the graph also contains a source node and a sink node: there are outgoing edges from the source to nodes 1 and $n/2 + 1$, and incoming edges to the sink from nodes $n/2$ and n.

After the first U requests, the adversary continues to generate requests, either in the left half of the line or the right half, depending on which half contains more than half of the bandwidth. For instance, if it is the left half, then the adversary introduces U new requests that can be routed either from node 1 to node $n/4$ or from node $n/4 + 1$ to node $n/2$. The adversary continues on with this strategy for $\log n$ rounds. We note that only $O(n)$ new source/sink nodes are added to the graph while generating the requests.

Next, consider an optimal solution that routes at most U requests on each edge. With this choice of paths there is a feasible bandwidth allocation that allocates $\log n$ requests with bandwidth 1. There is also a feasible solution that allocates bandwidth $1/2$ to $2\log n$ requests. In general, there is a feasible solution that allocates bandwidth of value $1/i$ to $i \log n$ requests, where $1 \leq i \leq U$. This means that an algorithm which is γ_c coordinate-wise competitive must allocate bandwidth of more than $1/\gamma_c$ to at least $\log n$ requests. In general, for any $1 \leq i \leq U$, the algorithm must give bandwidth at least $1/i\gamma_c$ to at least $i \log n$ requests. By this observation, the total bandwidth allocated by any γ_c-coordinate-wise online algorithm must be at least

$$\sum_{i=1}^{U} \frac{\log n}{i\gamma_c} \geq \frac{\log n \log U}{\gamma_c}.$$

By the adversary's strategy, there exists an edge such that at least half of the total bandwidth is routed on it. Since the total bandwidth allocated to this edge is at most 1, we get that

$$\gamma_c \geq \frac{\log n \log U}{2} = \Omega(\log n \log U). \qquad \square$$

9.3 Notes

The results in this chapter are based on the work of Buchbinder and Naor [33]. In this work, several routing algorithms (achieving various routing goals in several models) are designed using the generic algorithm. Other models studied in [33] are, for example, the fixed routes model and a model in which the algorithm is allowed to allocate weights to requests instead of actual bandwidths (see [33]).

Routing algorithms have been studied extensively in the literature. In [8, 11], two different (but similar in spirit) online routing algorithms were suggested. The objective in [11] is maximizing the total throughput, while the algorithm in [8] minimizes the load. Both of these algorithms can be recast within the primal–dual framework (see [33]).

The notion of *all-or-nothing* routing was defined in [39]. The elegant notion of *max–min fairness* was considered in many settings [23, 67]. The general framework of prefix and coordinate-wise competitiveness was suggested in [74]. More properties of these measures (in the offline case) were studied later on in [57, 77]. Goel et al. [59] studied the problem of achieving coordinate-wise competitiveness online. They designed an algorithm which is $O(1(\epsilon \log^2 n (\log U)^{1+\epsilon}))$-coordinate-wise competitive for any $\epsilon > 0$. In a more relaxed setting where the algorithm is allowed to assign weights instead of allocating bandwidth directly, Goel et al. [59] designed an algorithm which is $O(\log^2 n \log U)$-coordinate-wise competitive.

10

Maximizing Ad-Auctions Revenue

Maximizing the revenue of a seller in an online auction has recently received much attention and studied in many models and settings. In particular, the way search engine companies such as Microsoft, Google, and Yahoo! maximize their revenue out of selling ad-auctions has been studied extensively. In the search engine environment, advertisers link their ads to (search) keywords and provide a bid on the amount paid each time a user clicks on their ad. When users send queries to search engines, along with the (algorithmic) search results returned for each query, the search engine displays funded ads corresponding to *ad-auctions*. The ads are instantly sold, or allocated, to interested advertisers (*buyers*). The total revenue out of this fast growing market is currently billions of dollars. Thus, algorithmic ideas that can improve the allocation of ads, even by a small percentage, are crucial.

The ad-auctions problem is modeled as a generalization of the online bipartite matching problem. There is a set I of n buyers, each buyer i $(1 \leq i \leq n)$ has a known daily budget of $B(i)$. There is a set M of items of size m. In the online setting, the items arrive one-by-one and

upon arrival of item j, $1 \leq j \leq m$, each buyer provides a bid $b(i,j)$ for buying it. The online algorithm can *allocate* (or *sell*) the item to any one of the buyers. We distinguish between *integral* and *fractional* allocations. In an integral allocation, an item can only be allocated to a single buyer. In a fractional allocation, items can be fractionally allocated to several buyers; however, for each item the sum of the fractions allocated to buyers cannot exceed 1. The revenue received from each buyer is defined to be the minimum between the sum of the bids of the items allocated to a buyer (times the fraction allocated) and the total budget of the buyer. That is, buyers can never be charged by more than their total budget. The objective is to maximize the total revenue of the seller. Let $R_{\max} = \max_{i \in I, j \in M}\{b(i,j)/B(i)\}$ be the maximum ratio between a bid of any buyer and his total budget.

A linear programming formulation of the fractional (offline) ad-auctions problem appears in Figure 10.1. Let $y(i,j)$ denote the fraction of item j allocated to buyer i. The objective function is maximizing the total revenue. The first set of constraints guarantees that an item is not allocated (fractionally) more than once, i.e., the sum of the fractions of each item is at most 1. The second set of constraints guarantees that each buyer does not spend more than his budget. In the primal problem there is a variable $x(i)$ for each buyer i, a variable $z(j)$ for each item j, and the constraint $b(i,j)x(i) + z(j) \geq b(i,j)$ needs to be satisfied.

10.1 The Basic Algorithm

The basic algorithm for the online ad-auctions problem generates primal and dual solutions to the linear program of Figure 10.1.

Dual (Packing)	Primal (Covering)
maximize: $\sum_{j=1}^{m}\sum_{i=1}^{n} b(i,j)y(i,j)$	minimize : $\sum_{i=1}^{n} B(i)x(i) + \sum_{j=1}^{m} z(j)$
subject to:	subject to:
$\forall 1 \leq j \leq m$: $\sum_{i=1}^{n} y(i,j) \leq 1$	For each (i,j): $b(i,j)x(i) + z(j) \geq b(i,j)$
$\forall 1 \leq i \leq n$: $\sum_{j=1}^{m} b(i,j)y(i,j) \leq B(i)$	$\forall i,j$: $x(i), z(j) \geq 0$
$\forall i,j$: $y(i,j) \geq 0$	

Fig. 10.1 The fractional ad-auctions problem (dual) and the corresponding primal problem.

Allocation algorithm:

- Initially, for each buyer i, $x(i) \leftarrow 0$.
- Upon arrival of new item j, allocate it to buyer i maximizing $b(i,j)(1 - x(i))$.
- If $x(i) \geq 1$, then do nothing. Otherwise:
  (1) Charge the buyer the minimum between $b(i,j)$ and his remaining budget and set $y(i,j) \leftarrow 1$
  (2) $z(j) \leftarrow b(i,j)(1 - x(i))$.
  (3) $x(i) \leftarrow x(i)(1 + b(i,j)/B(i)) + b(i,j)/((c-1) \cdot B(i))$ (c is determined later).

Theorem 10.1. The allocation algorithm is $(1 - 1/c)(1 - R_{\max})$-competitive, where $c = (1 + R_{\max})^{1/R_{\max}}$. When $R_{\max} \to 0$, the competitive ratio tends to $(1 - 1/e)$.

Proof. Let P and D denote the values of the primal and dual solutions during the execution of the algorithm. We prove three simple claims:

(1) The algorithm produces a primal feasible solution.
(2) In each iteration, $\Delta P \leq (1 + 1/(c-1)) \cdot \Delta D$, where ΔP and ΔD are the changes to the values of the primal and dual objective functions.
(3) The algorithm produces an almost feasible dual solution.

Proof of (1): Consider a primal constraint corresponding to buyer i and item j. If $x(i) \geq 1$ then the primal constraint is satisfied. Otherwise, the algorithm allocates the item to the buyer i' for which $b(i',j)(1 - x(i'))$ is maximized. Setting $z(j) = b(i',j)(1 - x(i'))$ guarantees that the constraint is satisfied for all (i,j). Subsequent increases of the variables $x(i)$'s cannot make the solution infeasible.

Proof of (2): Whenever the algorithm updates the primal and dual solutions, the change in the dual profit is $b(i,j)$. (Note that even if the remaining budget of buyer i, to whom item j is allocated, is less than

his bid $b(i,j)$, variable $y(i,j)$ is still set to 1.) The change in the primal cost is

$$\begin{aligned} B(i)\Delta x(i) + z(j) &= \left(b(i,j)x(i) + \frac{b(i,j)}{c-1} \right) + b(i,j)(1 - x(i)) \\ &= b(i,j)\left(1 + \frac{1}{c-1}\right). \end{aligned}$$

Proof of (3): The algorithm never updates the dual solution for buyers satisfying $x(i) \geq 1$. We prove that for any buyer i, when $\sum_{j \in M} b(i,j)y(i,j) \geq B(i)$, then $x(i) \geq 1$. This is done by proving that:

$$x(i) \geq \frac{1}{c-1}\left(c^{\frac{\sum_{j \in M} b(i,j)y(i,j)}{B(i)}} - 1 \right). \tag{10.1}$$

Thus, whenever $\sum_{j \in M} b(i,j)y(i,j) \geq B(i)$, we get that $x(i) \geq 1$. We prove (10.1) by induction on the (relevant) iterations of the algorithm. Initially, this assumption is trivially true. We are only concerned with iterations in which an item, say k, is sold to buyer i. In such an iteration we get that:

$$\begin{aligned} x(i)_{\text{end}} &= x(i)_{\text{start}}\left(1 + \frac{b(i,k)}{B(i)}\right) + \frac{b(i,k)}{(c-1)B(i)} \\ &\geq \frac{1}{c-1}\left[c^{\frac{\sum_{j \in M \setminus \{k\}} b(i,j)y(i,j)}{B(i)}} - 1 \right]\left(1 + \frac{b(i,k)}{B(i)}\right) + \frac{b(i,k)}{(c-1)\cdot B(i)} \\ &= \frac{1}{c-1}\left[c^{\frac{\sum_{j \in M \setminus \{k\}} b(i,j)y(i,j)}{B(i)}}\left(1 + \frac{b(i,k)}{B(i)}\right) - 1 \right] \\ &\geq \frac{1}{c-1}\left[c^{\frac{\sum_{j \in M \setminus \{k\}} b(i,j)y(i,j)}{B(i)}} c^{\left(\frac{b(i,k)}{B(i)}\right)} - 1 \right] \\ &= \frac{1}{c-1}\left[c^{\frac{\sum_{j \in M} b(i,j)y(i,j)}{B(i)}} - 1 \right]. \end{aligned}$$

The second inequality follows from the induction hypothesis, and the one before last inequality follows since, for any $0 \leq x \leq y \leq 1$, $\ln(1+x)/x \geq \ln(1+y)/y$. Note that when $b(i,k)/B(i) = R_{\max}$ the latter inequality holds with equality. This is why we set the value of c to be

$(1 + R_{\max})^{\frac{1}{R_{\max}}}$. Thus, it follows that whenever the sum of charges to a buyer exceeds his budget, we stop charging this buyer. Hence, there can be at most one iteration in which a buyer is charged by less than $b(i,j)$. Therefore, for each buyer i:

$$\sum_{j \in M} b(i,j)y(i,j) \leq B(i) + \max_{j \in M}\{b(i,j)\},$$

and thus the profit extracted from buyer i is at least:

$$\left[\sum_{j \in M} b(i,j)y(i,j)\right] \frac{B(i)}{B(i) + \max_{j \in M}\{b(i,j)\}}$$
$$\geq \left[\sum_{j \in M} b(i,j)y(i,j)\right](1 - R_{\max}).$$

By the second claim the dual value is at least $1 - 1/c$ times the primal value, and thus (by weak duality) we conclude that the competitive ratio of the algorithm is $(1 - 1/c)(1 - R_{\max})$. □

10.2 Multiple Slots: The Role of Strong Duality

In this section, we consider a more general ad-auctions model in which there are several advertisement slots available. We show how to extend the allocation algorithm in a very elegant way using *strong duality*. Suppose there are ℓ slots to which ad-auctions can be allocated and suppose that buyers are allowed to provide bids on keywords which are slot-dependent. Denote the bid of buyer i on keyword j and slot k by $b(i,j,k)$. The restriction is that an (integral) allocation of a keyword to two different slots cannot be sold to the same buyer. The linear programming formulation of the problem appears in Figure 10.2. We are going to solve a maximum weight matching problem in a bipartite graph for each new keyword that appears. Our analysis of the competitive factor crucially relies on strong duality, i.e., the value of a maximum weight (integral) matching is equal to the value of an optimal primal solution. This, in a sense, allows us to extend the analysis of

Dual (Packing)	
maximize:	$\sum_{j=1}^{m}\sum_{i=1}^{n}\sum_{\ell=1}^{k} b(i,j,\ell)y(i,j,\ell)$
subject to:	
$\forall 1 \le j \le m,\ 1 \le k \le \ell$:	$\sum_{i=1}^{n} y(i,j,k) \le 1$
$\forall 1 \le i \le n$:	$\sum_{j=1}^{m}\sum_{k=1}^{\ell} b(i,j,k)y(i,j,k) \le B(i)$
$\forall 1 \le j \le m, 1 \le i \le n$:	$\sum_{k=1}^{\ell} y(i,j,k) \le 1$
Primal (Covering)	
minimize :	$\sum_{i=1}^{n} B(i)x(i) + \sum_{j=1}^{m}\sum_{k=1}^{\ell} z(j,k) + \sum_{i=1}^{n}\sum_{j=1}^{m} s(i,j)$
subject to:	
$\forall i,j,k$:	$b(i,j,k)x(i) + z(j,k) + s(i,j) \ge b(i,j,k)$

Fig. 10.2 The fractional multi-slot problem (dual) and the corresponding primal problem.

the single slot case to the multiple slot case in an almost "black box" fashion.

The algorithm for the online ad-auctions problem is as follows. Note that the algorithm does not update the variables $z(\cdot)$ and $s(\cdot)$ explicitly. These variables are only used for the purpose of analysis.

Allocation algorithm:

- Initially, for each buyer i, $x(i) \leftarrow 0$.
- Upon arrival of a new item j:
  - (1) Generate a bipartite graph H: n buyers on one side and ℓ slots on the other side. Edge $(i,k) \in H$ has weight $b(i,j,k)(1-x(i))$.
  - (2) Find a maximum weight (integral) matching in H, i.e., an assignment to the variables $y(i,j,k)$.
  - (3) Charge buyer i the minimum between $\sum_{k=1}^{\ell} b(i,j,k)y(i,j,k)$ and his remaining budget.
  - (4) For each buyer i, if there exists slot k for which $y(i,j,k) > 0$, then:

$$x(i) \leftarrow x(i)\left(1 + \frac{b(i,j,k)y(i,j,k)}{B(i)}\right) + \frac{b(i,j,k)y(i,j,k)}{(c-1)B(i)}.$$

Theorem 10.2. The algorithm is $(1 - 1/c)(1 - R_{\max})$-competitive, where c tends to e when $R_{\max} \to 0$.

Proof. We prove three simple claims:

(1) The algorithm produces a primal feasible solution.
(2) In each iteration, $\Delta P \leq (1 + 1/(c-1)) \cdot \Delta D$.
(3) The algorithm produces an almost feasible dual solution.

To prove the claims, we crucially use strong duality. The value of a maximum weight (integral) matching in H is equal to the value of an optimal primal solution. The primal and dual linear programs for the maximum weight matching problem appear in Figure 10.3. Note that in the dual problem, $x(i)$, $1 \leq i \leq n$, is now a constant. Further, note that the primal variables are actually the same as the primal variables of the multiple-slot problem (see Figure 10.2). By strong duality, the maximum weight matching algorithm outputs optimal primal and dual solutions satisfying:

$$\sum_{i=1}^{n}\sum_{k=1}^{\ell} b(i,j,k)\,(1-x(i))\,y(i,j,k) = \sum_{i=1}^{n} s(i,j) + \sum_{k=1}^{\ell} z(j,k).$$

Proof of (1): Recall that the primal constraint of the multiple-slot problem for buyer i, item j, and slot k (see Figure 10.2) is

$$\forall i,j,k: \; b(i,j,k)x(i) + z(j,k) + s(i,j) \geq b(i,j,k).$$

Since $z(j,k) + s(i,j) \geq b(i,j,k)((1-x(i))$, the above constraint is satisfied.

Dual (Packing)		Primal (Covering)	
maximize: $\sum_{i=1}^{n}\sum_{k=1}^{\ell} b(i,j,k)\,(1-x(i))\,y(i,j,k)$		minimize : $\sum_{i=1}^{n} s(i,j) + \sum_{k=1}^{\ell} z(j,k)$	
subject to:		subject to:	
$\forall 1 \leq k \leq \ell$:	$\sum_{i=1}^{n} y(i,j,k) \leq 1$	$\forall (i,k)$:	$s(i,j) + z(j,k) \geq b(i,j,k)\,(1-x(i))$
$\forall 1 \leq i \leq n$:	$\sum_{k=1}^{\ell} y(i,j,k) \leq 1$	$\forall i,k$:	$s(i,j), z(j,k) \geq 0$
$\forall i,k$:	$y(i,j,k) \geq 0$		

Fig. 10.3 The maximum weight matching problem for item j.

Proof of (2): Upon arrival of the jth item:

$$\begin{aligned}
\Delta P &= \sum_{i=1}^{n} z(j,i) + \sum_{k=1}^{\ell} s(j,i) + \sum_{i=1}^{n} B(i)\Delta x(i) \\
&= \sum_{i=1}^{n}\sum_{k=1}^{\ell} b(i,j,k)\,(1 - x(i))\,y(i,j,k) \\
&\quad + \sum_{i=1}^{n}\sum_{k=1}^{\ell} B(i)\left(\frac{b(i,j,k)x(i)y(i,j,k)}{B(i)} + \frac{b(i,j,k)y(i,j,k)}{(c-1)\cdot B(i)}\right) \\
&= \sum_{i=1}^{n}\sum_{k=1}^{\ell} b(i,j,k)y(i,j,k)\left(1 + \frac{1}{c-1}\right).
\end{aligned}$$

Since $\Delta D = \sum_{i=1}^{n}\sum_{k=1}^{\ell} b(i,j,k)y(i,j,k)$, the claim follows.

Proof of (3): The algorithm never updates the dual solution for buyers satisfying $x(i) \geq 1$. We prove that for any buyer i, when $\sum_{j=1}^{m}\sum_{k=1}^{\ell} b(i,j,k)y(i,j,k) \geq B(i)$, then $x(i) \geq 1$. This is done by showing that

$$x(i) \geq \frac{1}{c-1}\left(c^{\frac{\sum_{j=1}^{m}\sum_{k=1}^{\ell} b(i,j,k)y(i,j,k)}{B(i)}} - 1\right). \tag{10.2}$$

Thus, whenever $\sum_{j=1}^{m}\sum_{k=1}^{\ell} b(i,j,k)y(i,j,k) \geq B(i)$, we get that $x(i) \geq 1$. We prove (10.2) by induction on the (relevant) iterations of the algorithm. Initially, it is trivially true. We are only concerned about iterations in which the kth slot of item j is sold to buyer i. The proof is the same as the proof of Theorem 10.1 in the basic case and we omit it.

It follows from the above that whenever the sum of the charges to a buyer exceeds his budget, we stop charging this buyer. Thus, there can be at most one iteration in which we charge the buyer by less than $b(i,j,k)$. Therefore, for each buyer i:

$$\sum_{j\in M}\sum_{k=1}^{\ell} b(i,j,k)y(i,j,k) \leq B(i) + \max_{j\in M,k}\{b(i,j,k)\},$$

and thus the profit extracted from buyer i is at least:

$$\left[\sum_{j\in M} b(i,j,k)y(i,j,k)\right] \frac{B(i)}{B(i)+\max_{j\in M,k}\{b(i,j,k)\}}$$
$$\geq \left[\sum_{j\in M}\sum_{k=1}^{\ell} b(i,j,k)y(i,j,k)\right](1-R_{\max}).$$

By the second claim, the value of the dual is at least $1-1/c$ times the value of the primal, and thus we conclude that the competitive ratio of the algorithm is $(1-1/c)(1-R_{\max})$. □

10.3 Incorporating Stochastic information

Can the worst-case bound shown in the previous sections on the competitive factor be improved under certain (mild) stochastic assumptions? Let us assume that either stochastically, or in accordance with historical data, we know that a bidder is likely to spend a good fraction of his budget. We want to tweak the allocation algorithm so that under this assumption the worst-case bound on the competitive factor improves. As we tweak the algorithm it is likely that the bidder may change the fraction of the budget spent. So we propose to tweak the algorithm gradually until some steady state is reached, i.e., no more tweaking is required. Let $0 \leq g_i \leq 1$ be a lower bound on the fraction of the budget buyer i is going to spend in a steady state. We show that having this additional information allows us to improve the (worst-case) competitive ratio to $1-(1-g)/(e^{1-g})$, where $g = \min_{i\in I}\{g_i\}$.

The main idea behind the improved allocation algorithm is that if a buyer spends at least a g_i fraction of his budget, then it means that the primal variable $x(i)$ is going to be large at the end. Thus, the value of $z(j)$ can be lowered, providing in turn additional "money" that can be used to increase $x(i)$ faster. The main issue is to determine the value of $x(i)$ once the buyer has spent a g_i fraction of his budget. Denote this value by $x_s(i)$. We choose it so that after the buyer has spent a g_i fraction of his budget, $x(i) = x_s(i)$, and after having extracted all of his budget, $x(i) = 1$. In addition, we need the change in the primal cost to be the same with respect to the dual profit in iterations where we sell

the item to a buyer i who has not yet passed the spending threshold of g_i. The optimal choice of $x_s(i)$ turns out to be $g_i/(c^{1-g_i} - (1-g_i))$, and the growth function of the primal variable $x(i)$, as a function of the fraction of the budget spent, should be linear until the buyer has spent a g_i fraction of his budget, and exponential (as in the previous sections) from that point on. The modified algorithm is as follows:

Allocation algorithm:

- Initially, for each buyer i, $x(i) \leftarrow 0$.
- Upon arrival of a new item j: allocate it to the buyer i maximizing $b(i,j)(1 - \max\{x(i), x_s(i)\})$, where $x_s(i) = g_i/(c^{1-g_i} - (1-g_i))$ (c is determined later).
- If $x(i) \geq 1$, then do nothing. Otherwise:
  (1) Charge the buyer the minimum between $b(i,j)$ and his remaining budget.
  (2) $z(j) \leftarrow b(i,j)(1 - \max\{x(i), x_s(i)\})$.
  (3) $x(i) \leftarrow x(i) + \max\{x(i), x_s(i)\}b(i,j)/B(i) + x_s(i)b(i,j)/B(i)$.

Theorem 10.3. If each buyer spends at least a g_i fraction of his budget, then the algorithm is $(1 - (1-g)/c^{1-g})(1 - R_{\max})$-competitive, where $c = (1 + R_{\max})^{1/R_{\max}}$.

Proof. We first prove a more general claim regarding the final value of the primal variable $x(i)$ corresponding to buyer i. Let $x(i, \text{end})$ be the final (highest) value of $x(i)$ (upon termination). By our assumption, buyer i extracted at least a g_i fraction of his budget. Whenever we charge a buyer i for an item, and $x(i) < x_s(i)$, the algorithm updates:

$$x(i) \leftarrow x(i) + \frac{b(i,j)}{B(i)} \left(x_s(i) + \frac{1-g_i}{c^{1-g_i} - (1-g_i)} \right).$$

Thus, the final value of $x(i)$ is

$$\begin{aligned} x(i, \text{end}) &\geq g_i \left(x_s(i) + \frac{1-g_i}{c^{1-g_i} - (1-g_i)} \right) = g_i x_s(i) + (1-g_i)x_s(i) \\ &= x_s(i) \end{aligned} \tag{10.3}$$

We next prove three simple claims:

- The algorithm produces a primal feasible solution.
- In each iteration, $\Delta P \leq (1 + (1-g)/(c^{1-g} - (1-g)))\Delta D$.
- The algorithm produces an almost feasible dual solution.

Proof of (1): Consider the primal constraint corresponding to buyer i and item j. In order to make this constraint feasible, we need to set $z(j) \geq \max\{0, b(i,j)(1 - x(i,\text{end}))\}$. By Equation (10.3), $x(i,\text{end}) \geq x_s(i)$. Thus, when item j arrives, setting $z(j)$ to be $b(i,j)(1 - \max\{x(i), x_s(i)\}) \geq b(i,j)(1 - x(i,\text{end}))$ suffices to satisfy the constraint. Since the algorithm chooses the buyer i maximizing this value, and sets $z(j)$ accordingly, we get that the constraint is satisfied.

Proof of (2): Whenever the algorithm allocates item j to buyer i, the change in the dual profit is $b(i,j)$. (Note that even if the remaining budget of buyer i is less than his bid $b(i,j)$, variable $y(i,j)$ is still set to 1.) The change in the primal cost is

$$\begin{aligned} & B(i)\Delta x(i) + z(j) \\ & \quad = B(i)\left(\frac{b(i,j)\max\{x(i), x_s(i)\}}{B(i)} + \frac{b(i,j)}{B(i)}\frac{1-g_i}{c^{1-g_i} - (1-g_i)}\right) \\ & \qquad + b(i,j)(1 - \max\{x(i), x_s(i)\}) \\ & \quad = b(i,j)\left(1 + \frac{1-g_i}{c^{1-g_i} - (1-g_i)}\right) \;\leq\; b(i,j)\left(1 + \frac{1-g}{c^{1-g} - (1-g)}\right). \end{aligned}$$

Proof of (3): The algorithm never updates the dual solution for a buyer i satisfying $x(i) \geq 1$. We prove that for any buyer i, $x(i) \geq 1$, if $\sum_{j \in M} b(i,j)y(i,j) \geq B(i)$. This is done by proving that if buyer i extracted g_i' fraction of his budget (i.e., $\sum_{j \in M} b(i,j)y(i,j) = g_i' \cdot B(i)$) then:

$$x(i) \geq \begin{cases} g_i'\left[x_s(i) + \frac{1-g_i}{c^{1-g_i} - (1-g_i)}\right] & \text{if } g_i' \leq g_i, \\ x_s(i)c^{g_i' - g_i} + \frac{1-g_i}{c^{1-g_i} - (1-g_i)}[c^{g_i' - g_i} - 1] & \text{if } g_i' > g_i. \end{cases} \tag{10.4}$$

It is easy to check that when $g_i' = g_i$, the above two bounds coincide and are equal to $x_s(i)$. Thus, if the claim is correct, then whenever

buyer i extracts all his budget we get that:

$$\begin{aligned} x(i) &\geq x_s(i)c^{1-g_i} + \frac{1-g_i}{c^{1-g_i}-(1-g_i)}\left[c^{1-g_i}-1\right] \\ &= \frac{g_i}{c^{1-g_i}-(1-g_i)}c^{1-g_i} + \frac{1-g_i}{c^{1-g_i}-(1-g_i)}\left[c^{1-g_i}-1\right] \\ &= 1. \end{aligned}$$

We prove inequality (10.4) by induction on the (relevant) iterations of the algorithm. Initially, it is trivially true. We are only concerned about iterations in which an item, say k, is sold to buyer i. Let g_i' be the fraction of the budget buyer i spent before the current allocation, and let $g_i'' = g_i' + b(i,j)/B(i)$ be the fraction of the budget buyer i spends after the current allocation. In iterations in which $x(i) < x_s(i)$ we get from equality (10.3) that $g_i' < g_i$ and thus:

$$\begin{aligned} x(i)_{\text{end}} &= x(i)_{\text{start}} + x_s(i)\frac{b(i,k)}{B(i)} + \frac{b(i,k)}{B(i)}\frac{1-g_i}{c^{1-g_i}-(1-g_i)} \\ &\geq g_i'\left[x_s(i) + \frac{1-g_i}{c^{1-g_i}-(1-g_i)}\right] + x_s(i)\frac{b(i,k)}{B(i)} \\ &\quad + \frac{b(i,k)}{B(i)}\frac{1-g_i}{c^{1-g_i}-(1-g_i)} \\ &= g_i''\left[x_s(i) + \frac{1-g_i}{c^{1-g_i}-(1-g_i)}\right], \end{aligned}$$

where the second inequality follows from the induction hypothesis. We also remark here that if the fraction of the budget extracted from buyer i before the iteration is less than g_i, and the budget extracted after the iteration is strictly more than g_i, then it is possible to divide the cost $b(i,j)$ of the item into two costs: $b(i,j)_1 + b(i,j)_2 = b(i,j)$, such that the budget extracted after virtually selling $b(i,j)_1$ is exactly g_i. We virtually sell both items to buyer i and change $x(i)$ in two iterations. It is easy to verify that the change of $x(i)$ is the same as if this was done in a single iteration.

In iterations in which $x(i) \geq x_s(i)$ we get from equality (10.3) that $g'_i \geq g_i$, and thus:

$$\begin{aligned}
x(i)_{\text{end}} &= x(i)_{\text{start}} \left(1 + \frac{b(i,k)}{B(i)}\right) + \frac{b(i,k)}{B(i)} \frac{1-g_i}{c^{1-g_i} - (1-g_i)} \\
&\geq \left[x_s(i) c^{g'_i - g_i} + \frac{1-g_i}{c^{1-g_i} - (1-g_i)} [c^{g'_i - g_i} - 1]\right] \\
&\quad \times \left(1 + \frac{b(i,k)}{B(i)}\right) + \frac{b(i,k)}{B(i)} \frac{1-g_i}{c^{1-g_i} - (1-g_i)} \qquad (10.5) \\
&= x_s(i) c^{g'_i - g_i} \left(1 + \frac{b(i,k)}{B(i)}\right) \\
&\quad + \frac{1-g_i}{c^{1-g_i} - (1-g_i)} \left(c^{g'_i - g_i} \left(1 + \frac{b(i,k)}{B(i)}\right) - 1\right) \\
&\geq x_s(i) c^{g'_i - g_i} c^{\left(\frac{b(i,k)}{B(i)}\right)} \\
&\quad + \frac{1-g_i}{c^{1-g_i} - (1-g_i)} \left(c^{g'_i - g_i} c^{\left(\frac{b(i,k)}{B(i)}\right)} - 1\right) \qquad (10.6) \\
&= x_s(i) c^{g''_i - g_i} + \frac{1-g_i}{c^{1-g_i} - (1-g_i)} [c^{g''_i - g_i} - 1],
\end{aligned}$$

where inequality (10.5) follows from the induction hypothesis, and inequality (10.6) follows since for any $0 \leq x \leq y \leq 1$, $\ln(1+x)/x \geq \ln(1+y)/y$.

By the above, it follows that whenever the sum of the charges to a buyer is more than his budget, we stop charging this buyer. Thus, there can be at most one iteration in which we charge the buyer by less than $b(i,j)$. Therefore, for each buyer i:

$$\sum_{j \in M} b(i,j) y(i,j) \leq B(i) + \max_{j \in M} \{b(i,j)\},$$

and thus the profit extracted from buyer i is at least:

$$\begin{aligned}
&\left[\sum_{j \in M} b(i,j) y(i,j)\right] \frac{B(i)}{B(i) + \max_{j \in M} \{b(i,j)\}} \\
&\qquad \geq \left[\sum_{j \in M} b(i,j) y(i,j)\right] (1 - R_{\max}).
\end{aligned}$$

By the second claim the dual value is at least:

$$1 - \frac{1 - g_i}{c^{1-g_i}} \geq 1 - \frac{1 - g}{c^{1-g}}$$

times the primal value, and thus (by weak duality) we conclude that the competitive ratio of the algorithm is $(1 - R_{\max})(1 - (1 - g)/c^{1-g})$. □

10.4 Notes

The results in this chapter are based on the work of Buchbinder et al. [30]. The paper contains further extensions and variants of the basic model. The ad-auctions model studied here was originally introduced by Mehta et al. [82] and they also proposed a simple deterministic online $(1 - 1/e)$-competitive algorithm. The analysis of the algorithm uses the notion of trade-off revealing LP. The work of Mehta et al. [82] builds on previous work on online bipartite matching [73] and online b-matching [69]. The online b-matching problem is a special case of the online ad-auctions problem in which all buyers have a budget of b and the bids are either 0 or 1. In [69], a deterministic online algorithm is given for b-matching with competitive ratio tending to $(1 - 1/e)$ (from below) as b increases. For online matching (b-matching where $b = 1$) [73] designed a randomized algorithm which is $1 - 1/e$ competitive.

In general, maximizing the revenue of a seller in both offline and online settings has been studied extensively in many different models [7, 26, 27, 80].

11

Graph Optimization Problems

In this section, we describe applications of the primal–dual approach to a wide range of graph and network optimization problems, focusing on problems that arise in the study of *connectivity* and *cuts* in graphs. As in previous chapters, the first step is formulating the problem in hand as an (online) covering linear program, thus enabling the use of the generic algorithms described in Section 4.2. This yields a fractional solution for the problem. We then implement known offline rounding methods in an online fashion so as to obtain integral solutions. This part of the solution is problem dependent.

Connectivity and cut problems in graphs are defined on an input graph $G = (V, E)$ (directed or undirected), a cost function $c : E \to \mathbb{R}^+$, and a requirement function f (to be defined later). The goal is to find a minimum cost subgraph that satisfies the requirement function f. Our model is online; that is, the requirement function is not known in advance and it is given "step-by-step" to the algorithm, while the input graph is known in advance. We consider an online version of a class of problems which we call *generalized connectivity*. The requirement function is a set of demands of the form $D = (S, T)$, where S and T are subsets of vertices in the graph such that $S \cap T = \emptyset$. A feasible solution

is a set of edges, such that for each demand $D = (S,T)$ there is a path from a vertex in S to a vertex in T. Examples of problems belonging to this class are Steiner trees, generalized Steiner trees, and the group Steiner problem. Less obvious examples are the set cover problem and non-metric facility location.

Cut problems in graphs involve separating sets of vertices from each other. We concentrate on a family of cut problems which we call *generalized cuts*. The requirement function is a set of demands of the form $D = (S,T)$, where S and T are subsets of vertices in the graph such that $S \cap T = \emptyset$. A feasible solution is a set of edges that separates for each demand $D = (S,T)$, any two vertices $s \in S$ and $t \in T$. Examples of problems belonging to this class are the multiway cut problem and the multicut problem (see e.g., [90]).

There is a natural linear programming relaxation for the problems that we consider. For generalized connectivity problems, a feasible fractional solution associates a fractional weight (capacity) with each edge, such that for each demand $D = (S,T)$ a unit of flow can be sent from S to T, where the flow on each edge does not exceed its weight. For generalized cuts, a feasible fractional solution associates a fractional weight (length) with each edge, which we interpret as inducing a distance function. The constraint is that for each demand $D = (S,T)$, the distance between any two vertices $s \in S$ and $t \in T$ is at least 1. This linear programming relaxation is very useful for computing (offline) an approximate solution for many problems which fall in this category. (Refer to [90] for more details.) We note that fractional solutions have a motivation of their own in certain network design problems and bandwidth allocation problems (see, e.g., [84]).

11.1 Formulating the Problem

Let $G = (V,E)$ be a graph (directed or undirected) with cost function $c : E \to \mathbb{R}^+$ associated with the edge set E. Suppose further that there is a weight function (or capacity function) $w : E \to \mathbb{R}^+$ associated with the edge set E. The *cost* of w is defined to be $\sum_{e \in E} c_e w_e$.

Let $A \subset V$ and $B \subset V$ be subsets of V such that $A \cap B = \emptyset$. Let G' be the graph obtained from G by adding a super-source s connected

to all vertices in A and a super-sink t connected to all vertices in B. The edges from s to A are directed into A and have infinite weight, and the edges from B to t are directed into t and have infinite weight. We say that there is a flow from A to B of value α if there exists a valid flow function that sends α units of flow from s to t satisfying the capacity function w. The shortest path from A to B is defined to be the shortest path with respect to w from any vertex $u \in A$ to any vertex $v \in B$ (i.e., the minimal distance between any pair of vertices in A and B). A requirement function is a set of demands of the form $D_i = (S_i, T_i)$, $1 \leq i \leq k$, where $S_i \subset V$, $T_i \subset V$ and $S_i \cap T_i = \emptyset$.

We first define the *generalized connectivity* problem. The input for the problem is a graph $G = (V, E)$ with cost function $c : E \to \mathbb{R}^+$ and a requirement function. A feasible *integral* solution is an assignment of weights (capacities) w from $\{0, 1\}$ to E, such that for each demand $D_i = (S_i, T_i)$, $1 \leq i \leq k$, there is a flow from S_i to T_i of value at least 1. A feasible *fractional* solution is an assignment of weights (capacities) w from $[0, 1]$ to E, such that for each demand $D_i = (S_i, T_i)$, $1 \leq i \leq k$, there is a flow from S_i to T_i of value at least 1. We note that it suffices to satisfy the flow constraint for each demand (S_i, T_i) separately. The cost of a solution is defined to be the cost of w. It is possible to define an LP relaxation for the fractional offline problem. For each demand D_i, let $C(D_i)$ be the set of cuts that separate S_i from T_i. The LP formulation is as follows:

$$(P) : \min \sum_{e \in E} c_e w_e$$

subject to:

for all $i, 1 \leq i \leq k,$ and each cut

$$C \in C(D_i) : \sum_{e \in C} w_e \geq 1$$

for all $e \in E, w_e \geq 0$.

We next define the *generalized cuts* problem. The input for this problem is again a graph $G = (V, E)$ with cost function $c : E \to \mathbb{R}^+$ and a requirement function. A feasible *integral* solution is a set of edges $E' \subseteq E$ that separates, for each demand $D_i = (S_i, T_i)$, any two vertices $a \in S_i$ and $b \in T_i$. Alternatively, we can think of each edge $e \in E'$ as

having weight $w(e) = 1$. Thus, the weight function w induces a distance function on the graph such that the distance between vertices separated by E' is at least 1. A feasible *fractional* solution is an assignment of weights w from $[0,1]$ to E, such that for each demand $D_i = (S_i, T_i)$, $1 \leq i \leq k$, the distance induced by w between each $a \in S_i$ and $b \in T_i$ is at least 1. The cost of a solution is defined to be the cost of w. Again, it is possible to obtain a covering LP formulation for the generalized cuts problem. For each demand D_i let $P(D_i)$ be the set of paths between any two vertices in S_i and T_i. The LP formulation is as follows:

$$(P) : \min \sum_{e \in E} c_e w_e$$

subject to:

for all $i, 1 \leq i \leq k,$ and each cut

$$P \in P(D_i) : \sum_{e \in P} w_e \geq 1$$

for all $e \in E, w_e \geq 0$

In an online setting, the graph $G = (V, E)$ along with the cost function c is known to the algorithm (as well as to the adversary) in advance. The set of requests of the form $D_i = (S_i, T_i)$ is then given one-by-one to the algorithm in an online fashion. Upon arrival of a new demand, the algorithm can only satisfy it by increasing weights of edges in the graph. However, the algorithm is not allowed to decrease the weight of an edge. Thus, previous demands remain satisfied.

Online algorithm: It is not hard to see that the online setting of both the generalized connectivity problem and the generalized cuts problem belongs to the primal–dual framework. The linear programming formulation of both problems is a covering problem. In particular, whenever a new demand D_i arrives, the new set of constraints corresponding to the set of paths, or cuts, between S_i and T_i is added to the LP. Although this set may contain an exponential number of constraints, it is still possible to use the algorithms from Section 4 for solving the problem, since it suffices to either determine that a solution is feasible, or find an unsatisfied primal constraint. This can be easily done using a maximum flow computation from the set S_i to the set T_i for

the case of generalized connectivity, or a shortest path computation in the case of generalized cuts. We note that the basic algorithm can be modified so that the total number of update iterations is polynomial in the size of the graph. Thus, it is possible to compute online a monotonically increasing fractional solution which is $O(\log m)$-competitive. It is even possible to improve the factor to $O(\log |C_{\max}|)$, where $C_{\max}$ is the maximum cut in the graph in the case of generalized connectivity. In the case of generalized cuts, the competitive ratio can be improved to $O(\log d)$, where d is the diameter of the graph.

11.2 The Group Steiner Problem on Trees

In this section, we derive a randomized online integral solution to a special case of the generalized connectivity. This is done by converting an offline randomized rounding method into an online randomized rounding method. We demonstrate these ideas using the group Steiner problem on trees.

The *group Steiner tree* problem on a rooted tree is defined as follows. We are given a rooted tree $T = (V, E, r)$ with a non-negative cost function $c : E \to \mathbb{R}^+$ and groups $g_1, g_2, \ldots, g_k \subset V$. Let r denote the root of the tree T. The objective is to find a minimum cost rooted subtree T' that contains at least one vertex from each of the groups g_i, $1 \leq i \leq k$. That is, using the terminology of the generalized connectivity problem, each request is of the form (r, g_i). In the online setting of the group Steiner problem, the groups arrive one-by-one, and the algorithm has to choose additional edges to its solution upon arrival of a new request, so that the solution contains at least one vertex from the new group. Note that the online set cover problem is a special case of this problem.

The group Steiner tree problem has an $O(\log n \log k)$-approximation algorithm, where k is the number of groups and n is the number of leaves in the tree [56]. This approximation is based on a clever randomized rounding technique and analysis. We show here how to derive an online randomized rounding algorithm for the problem. This is done by basically imitating the offline randomized rounding method of [56].

The randomized rounding method we propose covers each group with probability $\Omega(1/\log N)$, where N is the maximum size of any

group. In addition, we prove that its expected cost is at most the cost of the fractional solution. We then run $O(\log k \log N)$ independent trials of this randomized rounding method in parallel, where k is the number of groups asked by the adversary. The algorithm takes to the solution each edge that was selected in any of the trials. Using simple probabilistic analysis we get that our algorithm has a competitive ratio of $O(\log n \log k \log N)$ and each of the groups is covered with probability at least $1 - 1/k$. In order to guarantee that the algorithm produces a feasible solution, we can use the shortest path to a group in case the algorithm fails to cover it. The cost of this path is certainly a lower bound on the optimal solution, and since this event happens with probability at most $1/k$, it changes the expected competitive ratio of the algorithm by only a negligible factor. Since we do not know in advance the value of k we can increase the number of trials gradually by doubling k whenever necessary.

Applying the technique of [56] requires that the fractional weights on a path from the root to any vertex are monotonically decreasing. However, the fractional solution that our algorithm computes may not necessarily satisfy this property. Therefore, we define the weight of each edge to be the maximum flow that can be routed through this edge to any vertex in its subtree. In the following, we abuse notation and let w_e denote the flow on edge e, instead of the actual weight of e. Since the flow routed on each edge is at most its weight, we note that this can only decrease the fractional value of the solution serving as our baseline for bounding the competitive analysis.

Our online randomized rounding method is the following. Initially, the algorithm starts with an empty cover $\mathbb{C} = \emptyset$. Consider an iteration in which the fractional weight of some edges is augmented. Since the weight of an edge is the maximum flow that can be routed through it, the fractional weight of an edge can be augmented when the algorithm augments the weights of other edges as well. If the weight of several edges is augmented at the same iteration, the rounding algorithm considers the edges one by one, according to a topological ordering, starting from the edges closer to the root. Let w_e and $w'_e = w_e + \delta_e$ be the weight of edge e before and after the weight augmentation, respectively. Let δ_e be the change in the weight of e. Let $e(p)$ be the edge adjacent to e

and closer to the root r. This definition is only relevant if the edge e is not incident on the root r. The rounding algorithm randomly chooses the edges to the solution by the following scheme.

Algorithm:
Consider all edges e for which $\delta_e > 0$ in any topological order:

- If $w'_e > 1$, add e to $\mathbb{C}$.
- If e is incident on r, or $w'_{e(p)} > 1$, add e to $\mathbb{C}$ with probability $\delta_e/(1 - w_e)$.
- If $e(p) \in \mathbb{C}$, add e to $\mathbb{C}$ with probability $\delta_e/(w'_{e(p)} - w_e)$.

Note that for each edge e that is not incident on the root, $\delta_e/(w'_{e(p)} - w_e) \leq \delta_e/(w'_e - w_e) = 1$, since $w'_e \leq w'_{e(p)}$. Thus, the probabilities are well defined. Furthermore, note that $\mathbb{C}$ induces a connected subtree of T. This follows since the edges that are augmented at the same iteration are considered in topological order and each edge may be added to $\mathbb{C}$ only if the path connecting it to the root r is already in $\mathbb{C}$. The proof of the performance of the algorithm is based on the simple observation that the online algorithm mimics the behavior of the corresponding offline algorithm in [56]. Thus, showing that the properties of the probability distribution that were needed for the analysis in the offline case are also maintained in the online case suffices. One of the tools used is induction on the steps of the algorithm. We state the main lemmas and omit the proofs.

Lemma 11.1. For each edge e, at the end of each iteration, the probability that $e \in \mathbb{C}$ is w'_e. If $w_e > 1$, then $e \in \mathbb{C}$ with probability 1.

The next lemma follows from linearity of expectation.

Lemma 11.2. At the end of each iteration, the expected cost of the edges in $\mathbb{C}$ is at most $\sum_{e \in \mathbb{T}} c_e w'_e$, where w'_e is the weight of edge e at the end of the iteration.

Let N be the maximum size of a group $g = \{\{v_1, v_2, \ldots, v_k\}$. Let w_g be the total flow that can be routed to the vertices in g simultaneously.

The next lemma bounds from below the probability that any group g with $w_g \geq 1$ is covered.

Lemma 11.3. In any iteration, if, for a group $g = \{v_1, v_2, \ldots, v_\ell\}$, $w_g \geq 1$, then the probability that there exists $v_i \in \mathbb{C}$ $(1 \leq i \leq \ell)$ is $\Omega(1/\log N)$.

To conclude, we state the performance of the randomized algorithm for the online group Steiner on trees.

Theorem 11.4. There is a randomized online algorithm for the group Steiner problem in trees with a competitive ratio of $O(\log^2 n \log k)$.

The group Steiner problem on general graphs: It is possible to combine the above theorem with the results of [46] on embedding a general metric (graph) in HSTs (similarly to what was done in the offline case). This gives the following result for the online group Steiner in general graphs.

Theorem 11.5. There is a randomized online algorithm for the group Steiner problem in general graphs with a competitive ratio of $O(\log^3 n \log k)$.

11.3 Notes

The results in this chapter are based on the work of Alon et al. [4]. The online fractional algorithm is analyzed in [4] via a potential function method rather than through the primal–dual approach. The paper contains several other problems that can be solved using the same approach.

The offline group Steiner was studied in [56]. They suggested an $O(\log n \log k)$-approximation algorithm for the problem on trees, where k is the number of groups and n is the number of leaves of the tree. For general undirected graphs the best approximation factor known for the group Steiner problem is $O(\log^2 n \log k)$ by combining [56]

with [46]. Online network optimization problems have been studied extensively. The online Steiner problem was considered in [63] who gave an $O(\log n)$-competitive algorithm and showed that in a general metric space this is indeed best possible. The online generalized Steiner problem was considered in [9], where an $O(\log^2 n)$-competitive algorithm was given. This bound was later improved to an $O(\log n)$-competitive algorithm by Berman and Coulston [22].

There is a vast literature on efficient (offline) approximation algorithms for problems involving connectivity and cuts. The reader is referred to [62, 90] for more details. In particular, the offline version of the generalized connectivity problem was considered in [37] who gave a polylogarithmic (offline) approximation to it.

12

Dynamic TCP-Acknowledgement Problem

In this section, we consider the dynamic TCP-acknowledgement problem, defined as follows. A source sends a stream of packets to a destination and it needs to receive an acknowledgement for each one of the packets. However, it is possible to acknowledge several packets by a single acknowledgement message. This can save on communication overhead, but requires delaying the acknowledgement of some of the packets (which is undesirable in general). For example, suppose the destination acknowledges the reception of packets $100, 101, \ldots, 120$ by a single message, then the acknowledgement of packets $100 + i$, $0 \leq i \leq 19$, is delayed till packet 120 arrives at the destination (or, possibly, even later). Two extreme solutions for the acknowledgement problem are acknowledging all packets by a single message, or acknowledging each packet by a separate message. Thus, there is a trade-off between the delay in acknowledging packets and the number of acknowledgement messages. The objective function is defined to be the number of acknowledgement messages sent plus the sum of the acknowledgement delays of the packets.

Dual (Packing)	Primal (Covering)
maximize: $\sum_{j \in M} \sum_{t \mid t \geq t(j)} y(j,t)$	minimize : $\sum_{t \in T} x_t + \sum_{j \in M} \sum_{t \mid t \geq t(j)} \frac{1}{d} z(j,t)$
subject to:	subject to:
$\forall t \in T$: $\sum_{j \mid t \geq t(j)} \sum_{t' \geq t} y(j,t') \leq 1$	$\forall j, t \mid t \geq t(j)$: $\sum_{k=t(j)}^{k=t} x_k + z(j,t) \geq 1$
$\forall j, t \mid t \geq t(j)$: $y(j,t) \leq \frac{1}{d}$	

Fig. 12.1 The fractional TCP problem (primal) and the corresponding dual problem.

Let M be the set of packets. For each packet $j \in M$, let $t(j)$ be the time of arrival at the destination. Assume now that packets can only arrive in discrete times of $1/d$. We later take $d \to \infty$ so this assumption is not going to be limiting. With the time discretizing assumption we can formulate the TCP-acknowledgement problem as a covering linear program which appears in Figure 12.1. In this formulation, we have a variable x_t for each discrete time t which is set to 1 if the algorithm sends an acknowledgement message at time t. For each packet j and time $t \geq t(j)$, we have a variable $z(j,t)$ which is set to 1 if packet j is delayed between time t and time $t + 1/d$. By this formulation, our objective is minimizing,

$$\sum_{t \in T} x_t + \sum_{j \in M} \sum_{t \mid t \geq t(j)} \frac{1}{d} z(j,t).$$

For each j and $\{t \mid t \geq t(j)\}$ we require that $\sum_{k=t(j)}^{t} x_k + z(j,t) \geq 1$. This guarantees that either the packet is delayed between time t and time $t + 1/d$, or an acknowledgement message was sent to the source since the arrival time of the packet. The dual packing problem has variables $y(j,t)$ for each packet j and $t \geq t(j)$.

12.1 The Algorithm

The covering linear programming formulation of the dynamic TCP-acknowledgment problem allows for the design of a simple primal–dual based algorithm for the problem. The algorithm is similar in spirit to the online algorithm presented for the ski rental in Section 3.

Dynamic TCP-acknowledgement algorithm:

- Initially, $\forall k,\ x_k \leftarrow 0$.
- At each discrete time t (iteration), consider each of the packets j for which $\sum_{k=t(j)}^{k=t} x_k < 1$.
- For each such packet j do the following update:
  (1) $z(j,t) \leftarrow 1 - \sum_{k=t(j)}^{k=t} x_k$
  (2) $x_t \leftarrow x_t + 1/d \sum_{k=t(j)}^{k=t} x_k + 1/((c-1)\cdot d)$ (c is chosen later).
  (3) $y(j,t) \leftarrow 1/d$.

The analysis is not very difficult: First, the primal solution we produce is feasible. This follows since we update for each unsatisfied packet $z(j,t) \leftarrow 1 - \sum_{k=t(j)}^{k=t} x_k$ for each time t.

The second observation is that for each packet j and time t in which we the variables are updated, the change in the dual profit is $1/d$, while the change in our primal cost is:

$$\left(1 - \sum_{k=t(j)}^{k=t} x_k\right)\frac{1}{d} + \frac{1}{d}\left(\sum_{k=t(j)}^{k=t} x_k + \frac{1}{c-1}\right) = \frac{1}{d}\left(1 + \frac{1}{c-1}\right).$$

Finally, we choose the parameter c to guarantee dual feasiblity. Consider a time t and a corresponding dual constraint

$$\sum_{j \mid t \geq t(j)} \sum_{t' \geq t} y(j,t') \leq 1.$$

We want to guarantee that after d updates of a variable $y(j,t')$ "belonging" to the constraint, all packets that arrived not later than t are satisfied, and therefore there are no more updates of variables belonging to the constraint. We prove that after d such updates, $\sum_{k \geq t} x_k \geq 1$, and therefore packets that have arrived prior to time t are satisfied.

We prove by induction on the update steps that

$$\sum_{k \geq t} x_k \geq \frac{(1+1/d)^q - 1}{c-1},$$

where q is the number of updates. Before the first update, the claim trivially holds. Consider an update of $y(j,t')$ (at time t') such that $t' \geq t$

and packet j arrived not later than t. From the algorithm we get:

$$x_{t'} \leftarrow x_{t'} + \frac{1}{d}\sum_{k=t(j)}^{k=t'} x_k + \frac{1}{(c-1)d} \geq x_{t'} + \frac{1}{d}\sum_{k=t}^{k=t'} x_k + \frac{1}{(c-1)d}$$

Therefore, $\sum_{k\geq t} x_k$ satisfies:

$$\begin{aligned}
&\left(1+\frac{1}{d}\right)\sum_{k\geq t} x_k + \frac{1}{(c-1)d} \\
&\qquad \geq \left(1+\frac{1}{d}\right)\frac{(1+1/d)^{q-1}-1}{c-1} + \frac{1}{(c-1)d} \\
&\qquad = \frac{(1+1/d)^q - 1}{c-1},
\end{aligned}$$

where the inequality follows by the induction hypothesis. Thus, choosing $c = (1+1/d)^d$ suffices, and if $d \to \infty$ we get a $(1-1/e)$-competitive algorithm.

In order to get a randomized integral solution we arrange the variables x_t on the infinite line. We choose a random number $p \in_R [0,1]$. We then send an acknowledgement message at each time segment x_t that falls in $p+k$ for some integer value k. We remark that we need the random choices to be correlated. It can be verified that our expected cost is the same as the cost of our fractional algorithm, completing the analysis.

12.2 Notes

The results in this chapter are based on the work of Buchbinder et al. [30]. The TCP-acknowledgment problem was introduced by Dooly et al. [45] who gave a 2-competitive algorithm for the problem. This bound was later improved by [70] to a randomized $(1-1/e)$-competitive algorithm. Our algorithm is an alternative primal–dual view of this algorithm. Buchbinder et al. [31] studied an online inventory problem which is a variant of the classical joint replenishment problem (JRP) that has been studied extensively over the years. This inventory problem is actually a generalization of the dynamic TCP-acknowledgment. In [31], a deterministic 3-competitive algorithm is given for the problem which is also based on a primal–dual approach.

13

The Bounded Allocation Problem: Beating $(1 - 1/e)$

We consider here the *allocation problem*, a special case of the ad-auctions problem that was studied in Chapter 10. In the allocation problem, a seller is interested in selling items to a group of buyers, where buyer i has budget $B(i)$. The seller introduces the items one-by-one and sets for each new item j a *fixed price* $b(j)$.[1] Each buyer then announces whether he is interested in buying the item for the set price. The seller then decides (instantly) to which of the interested buyers to sell the item. There is a lower bound example showing that without further assumptions any algorithm for the problem has competitive ratio of at most $1 - 1/e$ [73]. This lower bound uses an instance in which the number of buyers interested in each item may be as large as the total number of buyers. However, in many realistic settings, for each item the set of interested buyers is much smaller than the total number of buyers. The question is whether we can take advantage of this fact to improve on the competitiveness of the algorithm. We answer this question in the affirmative. The main novel idea we demonstrate here is a non-intuitive fractional algorithm for the problem.

[1] Note that in the ad-auctions problem the price is not fixed for all buyers.

Table 13.1 Upper and lower bounds on the competitive ratio for some values of d.

	Lower bound	Upper bound		Lower bound	Upper bound
$d = 2$	0.75	0.75	$d = 10$	0.662	0.651
$d = 3$	0.704	0.704	$d = 20$	0.648	0.641
$d = 5$	0.686	0.672	$d \to \infty$	0.6321...	0.6321...

For each item j let $S(j)$ be the set of interested buyers. We assume that there is an upper bound d such that for each item j, $|S(j)| \leq d$. We are interested in the case in which $d \ll n$, where n is the total number of buyers. We design an online algorithm with competitive ratio $C(d) = 1 - (d-1)/(d(1+1/(d-1))^{d-1})$. This factor is strictly better than $1 - 1/e$ for any value of d, and approaches $(1 - 1/e)$ from above as d goes to infinity. We also prove lower bounds for the problem that indicate that the competitive factor of the online algorithm is quite tight. The improved bounds for certain values of d are shown in Table 13.1.

13.1 The Algorithm

The first step is to cast the problem as a linear program using the same formulation as in Section 10. Let $y(i,j)$ be an indicator to the event that item j is allocated to buyer i. Then the offline problem can be cast as the dual linear formulation in Figure 13.1.

In a fractional solution to the problem, each item is sold fractionally to several buyers. The fractional version of the problem has a motivation of its own, e.g., in case items can be divided between buyers. An example of a divisible item is the allocation of bandwidth in a communication network. The fractional algorithm we design is somewhat counter-intuitive. In particular, a newly arrived item is not split equally

Dual (Packing)	Primal (Covering)
maximize: $\sum_{j=1}^{m} \sum_{i \in S(j)} b(j)y(i,j)$	minimize : $\sum_{i=1}^{n} B(i)x(i) + \sum_{j=1}^{m} z(j)$
subject to:	subject to:
$\forall 1 \leq j \leq m$: $\sum_{i \in S(j)} y(i,j) \leq 1$	For each $j, i \in S(j)$: $b(j)x(i) + z(j) \geq b(j)$
$\forall 1 \leq i \leq n$: $\sum_{j \mid i \in S(j)} b(j)y(i,j) \leq B(i)$	$\forall i,j$: $x(i), z(j) \geq 0$
$\forall i,j$: $y(i,j) \geq 0$	

Fig. 13.1 The fractional allocation problem (dual) and the corresponding primal problem.

between buyers who have spent the least fraction of their budget. Such an algorithm is referred to in the literature as a "water level" algorithm and it is not hard to verify that it does not improve upon the $(1 - 1/e)$ worst case ratio, even for small values of d. Rather, the idea is to split the item between several buyers that have *approximately* spent the same fraction of their total budget.

We divide the buyers into *levels* according to the fraction of the budget already spent. For $0 \leq k \leq d$, let $L(k)$ be the set of buyers that have spent at least a fraction of k/d and less than a fraction of $(k+1)/d$ of their budget (buyers in level d have exhausted their budget). We refer to each $L(k)$ as level k and say that it is *non-empty* if it contains buyers. The formal description of the algorithm is as follows:

Allocation algorithm: Upon arrival of a new item j, allocate the item to the buyers according to the following rules:

- Allocate the item equally and continuously between interested buyers in the lowest non-empty level containing buyers from $S(j)$. If, during the allocation, some of the buyers have moved to a higher level, then continue to allocate the item equally only among the buyers in the lowest level.
- If all interested buyers in the lowest level have moved to a higher level, then allocate the remaining fraction of the item equally and continuously between the buyers in the new lowest level. If all interested buyers have exhausted their budget, then stop allocating the remaining fraction of the item.

For the analysis, we find the best trade-off function, f_d, relating the values of the primal variables to the value of the dual objective function. The best function turns out to be a piecewise linear function consisting of d linear segments. As d grows, the function approximates the exponential function $f_{(d=\infty)}(x) = (e^x - 1)/(e - 1)$. The linear pieces are obtained from a geometric sequence a_t $(1 \leq t \leq d)$, defined inductively

as follows:

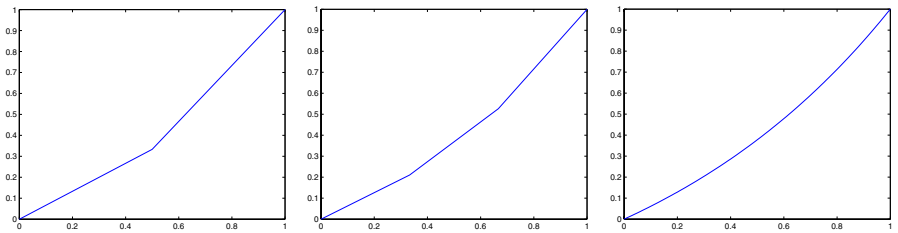
$$a_1 = \frac{1}{d\left(1 + \frac{1}{d-1}\right)^{d-1} - (d-1)}, \ldots, a_t = a_1 \left(1 + \frac{1}{d-1}\right)^{t-1}.$$

The sequence a_t is a geometric sequence and we only consider the first d elements in the sequence. The potential function f_d is defined for any $0 \leq j \leq d$ to be $f_d(j/d) \triangleq \sum_{t=1}^{j} a_t$. A simple calculation yields the following, for any j, $1 \leq j \leq d$:

$$\begin{aligned} f_d\left(\frac{j}{d}\right) &= \sum_{i=1}^{j} a_i = a_1 \frac{\left(1 + \frac{1}{d-1}\right)^j - 1}{(1 + \frac{1}{d-1}) - 1} \\ &= a_1 \left[d\left(1 + \frac{1}{d-1}\right)^{j-1} - (d-1) \right]. \end{aligned}$$

In particular, setting $j = d$, we get $f_d(d/d) = 1$. This piecewise linear approximation allows us to better analyze the algorithm and improve the competitive factors. The function f_d appears in Figure 13.2 for $d = 2$, $d = 3$, and d tending to infinity. Next, we use the potential function to prove that the allocation algorithm has the desired competitive factor with respect to an optimal offline fractional solution.

Theorem 13.1. The allocation algorithm is $C(d)$-competitive, where $C(d) = 1 - (d-1)/(d(1 + 1/(d-1))^{d-1})$.



$d = 2$ $\quad$ $d = 3$ $\quad$ $d \to \infty$

Fig. 13.2 The function f_d for $d = 2$ and $d = 3$. The y middle value for $d = 2$ is 1/3. The y middle values for $d = 3$ are 4/19 and 10/19.

Proof. Let $Y(j)$ denote the total profit of the algorithm (the dual packing) in the jth iteration. In each iteration, we maintain a corresponding feasible primal solution whose value is denoted by $X(j)$. Upon arrival of a new item we update both primal and dual programs. The dual (packing) program is updated by adding a new constraint corresponding to the new item which has arrived, and by adding a new term $b(j)y(i,j)$ to each constraint of an interested buyer. The primal program is updated by adding a new variable $z(j)$ for the new item and a constraint of the form $b(j)x(i) + z(j) \geq b(j)$ for each interested buyer.

Initially, the dual and primal programs are empty. In the jth iteration, the change in the values of the primal and dual solutions is denoted by $\Delta X(j)$ and $\Delta Y(j)$, correspondingly. We prove that in each iteration:

$$\Delta X(j) \leq \frac{1}{C(d)} \Delta Y(j).$$

The primal solution is an assignment of values to the variables $x(i)$ and $z(j)$. Since these values are not used by the allocation algorithm, we can set them using future knowledge. For each buyer i, let $t(i)$ $(0 \leq t \leq d)$ be the highest level i to which this buyer belongs during the execution of the algorithm. Thus, buyer i spent overall at least $t(i)/d$ fraction of his budget. The variable $x(i)$ grows as a function of the fraction of money that buyer i spent, which in fact depends on the corresponding dual constraint. Specifically, for buyer i:

$$x(i) = \begin{cases} f_d\left(\frac{1}{B(i)} \sum_{j \mid i \in S(j)} b(j)y(i,j)\right) & \text{if } \frac{1}{B(i)} \sum_{j \mid i \in S(j)} b(j)y(i,j) \leq \frac{t(i)}{d}, \\ f_d\left(\frac{t(i)}{d}\right) & \text{if } \frac{1}{B(i)} \sum_{j \mid i \in S(j)} b(j)y(i,j) \geq \frac{t(i)}{d}. \end{cases}$$

The variables $x(i)$ are monotonically increasing and thus, once a primal constraint is satisfied, it remains satisfied throughout the run of the algorithm. Hence, in each iteration, it suffices to satisfy the newly added primal constraints.

Consider first the case in which item j was not fully sold by the algorithm. This means that at the end of the jth iteration all the buyers

in $S(j)$ exhausted their budget. In this case, the corresponding variables $x(i)$ at the end of the iteration are all 1, and thus all the new primal constraints are satisfied, and we can set $z(j) = 0$. We only need to show that the change in the primal profit in this iteration is not too large. When we increase a variable $y(i,j)$, the derivative of the dual profit of the algorithm is $b(j)$. The derivative of the primal cost is

$$B(i)\frac{\mathrm{d}f_d}{\mathrm{d}(y(i,j))} \leq B(i)\frac{b(j)}{B(i)}da_d = \frac{1}{C(d)}b(j).$$

The inequality follows by taking the maximum derivative of the (convex) function f_d which is

$$da_d = da_1\left(1 + \frac{1}{d-1}\right)^{d-1} = \frac{1}{C(d)}.$$

Thus, we get that in this iteration $\Delta X(j) \leq (1/C(d))\Delta Y(j)$.

Assume now that item j was fully sold to the buyers. Let t, $0 \leq t \leq d-1$, be the highest level of buyers to which the item was sold. Since the algorithm always allocates the item to buyers in the lowest possible level, it means that all buyers in $S(j)$ used at least t/d fraction of their money. Let $\Delta_0, \Delta_1, \ldots, \Delta_t$ be the fraction of the item that was allocated in each level $k \leq t$. By our assumption, $\sum_{k=1}^{t}\Delta_k = 1$. We consider two cases.

Case 1: All the buyers in $S(j)$ spend during the algorithm at least t'/d of their budget for $t' > t$. In this case, for each buyer i, the derivative of the primal cost due to the change in $x(i)$ is

$$B(i)\frac{\mathrm{d}f_d}{\mathrm{d}(y(i,j))} \leq B(i)\frac{b(j)}{B(i)}da_{t+1} = b(j)da_{t+1}.$$

The inequality follows by taking the derivative of f_d in the highest level in which the item is sold. We fully allocate the item and hence $\sum_{i\in S(j)} y(i,j) = 1$. Thus, the total change of the primal cost due to the change in the variables $x(i)$ is at most $b(j)da_{t+1}$. Since all buyers in $S(j)$ eventually spend during the algorithm at least $(t+1)/d$ of their budget, variable $x(i)$ corresponding to buyer $i \in S(j)$ will be at the end of the allocation process at least $f((t+1)/d)$. Therefore, it is safe to

set $z(j) = b(j)(1 - f((t+1)/d))$ in order to satisfy all the new primal constraints. Thus, the total change in the primal cost in this iteration is:

$$\begin{aligned}
&z(j) + \sum_{i \in S'(j)} B(i)\Delta(x(i)) \\
&\quad \leq b(j)\left(1 - f(\frac{t+1}{d})\right) + b(j)da_{t+1} \\
&\quad = b(j)\left(1 - a_1\left[d\left(1 + \frac{1}{d-1}\right)^t - (d-1)\right]\right) \\
&\qquad + b(j)da_1\left(1 + \frac{1}{d-1}\right)^t \\
&\quad = b(j)\left(1 + a_1(d-1)\right) \\
&\quad = b(j)\left(1 + \frac{d-1}{d\left(1 + \frac{1}{d-1}\right)^{d-1} - (d-1)}\right) \\
&\quad = \frac{1}{C(d)}b(j).
\end{aligned}$$

Since the item was fully sold the dual profit in this case is $b(j)$ and hence we are done with this case.

Case 2: There exists at least one buyer in $S(j)$ who eventually spends (throughout the algorithm) less than a fraction of $(t+1)/d$ of his budget (but spends at least t/d). In this case, in order to satisfy the new primal constraint, it is only safe to set $z(j) = b(j)(1 - f(t/d))$. However, note that the buyer that spent less than $(t+1)/d$ fraction of his money was present throughout the whole process of dividing the item equally between all buyers in the last level t. Thus, by our algorithm, this buyer receives at least a fraction of Δ_t/d of the item. By the definition of the function associated with the variable $x(i)$, the growth function of $x(i)$ in this segment (which is larger than $t(i)$) is zero. Thus, the change in the primal cost due to the increase of the dual variables in the highest level is at most:

$$b(j)a_{t+1}d\frac{d-1}{d}\Delta_t = b(j)(d-1)a_{t+1}\Delta_t. \tag{13.1}$$

The change in the primal cost due to the increase of the dual variables in lower levels is at most $b(j)da_t(1-\Delta_t)$. But, $a_{t+1} = a_t(1 + 1/(d-1))$, and so $a_t = ((d-1)/d)a_{t+1}$. Thus, the change in the primal cost due to change in the variables $x(i)$ is at most:

$$\begin{aligned} b(j)da_t(1-\Delta_t) &= b(j)d\frac{d-1}{d}a_{t+1}(1-\Delta_t) \\ &= b(j)(d-1)a_{t+1}(1-\Delta_t). \end{aligned} \tag{13.2}$$

Adding up Equations (13.1) and (13.2), we get that the total change in the primal cost due to the increase in the primal variables $x(i)$ is $b(j)(d-1)a_{t+1}$. Since $f((t+1)/d) = a_{t+1} + f(t/d)$, the total change in the primal cost is at most:

$$\begin{aligned} & b(j)\left(1 - f\left(\frac{t}{d}\right)\right) + b(j)(d-1)a_{t+1} \\ &= b(j)\left(1 - f\left(\frac{t+1}{d}\right) + a_{t+1})\right) + b(j)(d-1)a_{t+1} \\ &= b(j)\left(1 - f\left(\frac{t+1}{d}\right)\right) + b(j)da_{t+1} \\ &= \frac{1}{C(d)}b(j). \end{aligned}$$

This change is exactly the same as in case 1. Similarly to case 1, the item was fully sold and so the dual profit is $b(j)$ and we are done with this case. □

Lower bounds: For any value of d it is not hard to prove the following lower bound.

Lemma 13.2. Let $H(\cdot)$ denote the harmonic number let k be the largest value for which $H(d) - H(d-k) \leq 1$. Then, for any d,

$$C(d) \leq 1 - \frac{k - kH(d) + \sum_{i=1}^{k} H(d-i)}{d}.$$

This general bound is only tight for $d = 2$, but we note that one can derive better tailor-made lower bounds for specific values of d. In particular, it is not hard to show that the algorithm is optimal for $d = 3$.

Rounding the fractional solution: It is possible to apply standard randomized rounding techniques in an online fashion. The main issue is that when applying randomized rounding the algorithm may allocate to buyer i items with total value of more than $B(i)$. However, using standard techniques one can prove that when the budget of each buyer is much larger than the price of the individual items, then with high probability the budget excess is not going to be large, i.e., the additional loss in the competitive factor is $o(1)$. In this case, it is also possible to apply de-randomization methods to the randomized rounding algorithm to obtain a deterministic algorithm for the problem.

13.2 Notes

The results in this section are based on the work of Buchbinder et al. [30]. They also showed how to de-randomize the algorithm to obtain a deterministic algorithm for the problem. The same technique can be used to improve the competitive ratio for other problems. In the ski rental problem, for example, one can obtain using this method an algorithm with improved competitive factor of $C(B)$, where B is the cost of buying the skis. In the dynamic TCP-acknowledgment problem studied in Section 12, it is also possible to improve the competitive ratio in certain scenarios. If packets only arrive in discrete times of $1/d$ (and not in any continuous time) then the competitive ratio can be improved to $C(d)$.

14

Extension to General Packing–Covering Constraints

In this section, we design primal–dual algorithms for more general settings of packing–covering linear formulations. In the more general (fractional) covering problem, the objective is still to minimize the total cost given by a linear cost function $\sum_{i=1}^{n} c(i)x(i)$. However, the feasible solution space is defined by a set of m linear constraints of the general form $\sum_{i=1}^{n} a(i,j)x(i) \geq b(j)$, where the entries $a(i,j)$ and $b(j)$ are non-negative. This generalizes the setting of Section 4 in which $a(i,j) \in \{0,1\}$ and $b(j) = 1$. Given an instance of a covering problem we can always normalize each constraint to the form: $\sum_{i=1}^{n} a(i,j)x(i) \geq 1$. Any primal covering instance has a corresponding dual packing problem that provides a lower bound on any feasible solution to the instance. A general form of a (normalized) primal covering problem along with its (normalized) dual packing problem is given in Figure 14.1. Throughout this chapter we refer to the covering problem as the "primal problem" and the packing problem as the "dual problem."

The online setting we study here is the same as the one studied in Section 4. In the general *online fractional covering problem*, the cost function is known in advance, but the linear constraints that define the feasible solution space are given to the algorithm one-by-one. Again we are only allowed to increase the variables, but not to decrease any

Primal (Covering)	Dual (Packing)
minimize: $\sum_{i=1}^{n} c(i)x(i)$	maximize: $\sum_{j=1}^{m} y(j)$
subject to:	subject to:
$\forall 1 \leq j \leq m$: $\sum_{i=1}^{n} a(i,j)x(i) \geq 1$	$\forall 1 \leq i \leq n$: $\sum_{j=1}^{m} a(i,j)y(j) \leq c(i)$
$\forall 1 \leq i \leq n$: $x(i) \geq 0$	$\forall 1 \leq j \leq m$: $y(j) \geq 0$

Fig. 14.1 Primal (covering) and dual (packing) problems.

previously increased variable. In the general *online fractional packing problem*, the values $c(i)$ $(1 \leq i \leq n)$ are known in advance. However, the profit function and the exact packing constraints are not known in advance. In the jth round, a new variable $y(j)$ is introduced to the algorithm, along with its set of coefficients $a(i,j)$ $(1 \leq i \leq n)$.[1] The algorithm can increase the value of a variable $y(j)$ only in the round in which it is given, and cannot decrease or increase the values of any previously given variables.

For this more general scenario, we need to design two different schemes. One scheme for the packing setting and a different one for the covering setting. The performance of the schemes is also different. For the packing setting, we design a scheme which is B-competitive for any $B > 0$ and for each constraint it holds that

$$\sum_{k=1}^{m} a(i,k)y(k) = c(i)O\left(\frac{\log n + \log \frac{a_i(\max)}{a_i(\min)}}{B}\right),$$

where $a_i(\min) = \min_{k=1}^{m}\{a(i,k)|a(i,k) \neq 0\}$ and $a_i(\max) = \max_{k=1}^{m}\{a(i,k)\}$. That is, there is an additional penalty of $\log(a_i(\max))/(a_i(\min))$. We prove that this additional penalty is unavoidable. For the covering setting, we design a scheme which is $O(\log n/B)$-competitive for any $B > 0$ and for each constraint $\sum_{i=1}^{n} a(i,j)x(i) \geq 1/B$. Thus, in this setting, there is no additional loss.

14.1 The General Online Fractional Packing Problem

In this section, we describe an online scheme for computing a near-optimal fractional solution for the general online fractional packing

[1] $y(j)$ can always be normalized so that its coefficient in the objective function is 1.

problem. The scheme gets the desired competitive ratio $B > 0$ and returns a solution which is within a factor of B of the optimal, and which does not violate the packing constraints by too much (to be made more precise shortly). We prove that the scheme is optimal up to constant factors. Our scheme simultaneously maintains primal (covering) and dual (packing) solutions for the primal and dual instances.

Initially, each variable $x(i)$ is initialized to zero. In each round, a new variable $y(j)$ is introduced along with its coefficients $a(i,j)$ $(1 \leq i \leq n)$. In the corresponding primal sub-instance, a new constraint is introduced of the form $\sum_{i=1}^{n} a(i,j)x(i) \geq 1$. Without loss of generality, we can assume that this constraint has at least one non-zero coefficient, otherwise it means that there is no bound on the value of $y(j)$ and the profit function is unbounded. The algorithm increases the value of the new variable $y(j)$ and the values of the primal variables $x(i)$ until the new primal constraint is satisfied. The augmentation method is described here in a continuous fashion, but it is not hard to implement the augmentation in a discrete way in any desired accuracy. In our continuous description, the variables $x(i)$ behave according to a monotonically increasing function of $y(j)$. To implement the scheme in a discrete fashion, one should find the minimum $y(j)$ such that the new primal constraint is satisfied. Note that variable $y(j)$ is being increased only in the jth round and the values of the primal variables never decrease. In the following, we describe the jth round. The performance of the scheme is analyzed in Theorem 14.1.

(1) $y(j) \leftarrow 0$; For each $x(i)$: $a_i(\max) \leftarrow \max_{k=1}^{j}\{a(i,k)\}$.
(2) While $\sum_{i=1}^{n} a(i,j)x(i) < 1$:
    (a) Increase $y(j)$ continuously.
    (b) Increase each variable $x(i)$ by the following increment function:

$$x(i) \leftarrow \max\left\{x(i), \frac{1}{na_i(\max)}\left[\exp\left(\frac{B}{2c(i)}\sum_{k=1}^{j} a(i,k)y(k)\right) - 1\right]\right\}.$$

Theorem 14.1. For any $B > 0$, the above scheme is a B-competitive algorithm for the general online fractional packing problem. Also, for each constraint it holds;

$$\sum_{k=1}^{m} a(i,k)y(k) = c(i)O\left(\frac{\log n + \log \frac{a_i(\max)}{a_i(\min)}}{B}\right),$$

where $a_i(\min) = \min_{k=1}^{m}\{a(i,k)|a(i,k) \neq 0\}$ and $a_i(\max) = \max_{k=1}^{m}\{a(i,k)\}$.

Proof. Let $X(j)$ and $Y(j)$ be the values of the primal and dual solutions, respectively, obtained in round j. We prove the following claims on $X(j)$ and $Y(j)$:

(1) In each round j: $Y(j) \geq X(j)/B$.
(2) The primal solution produced by the scheme is feasible.
(3) For any dual constraint:

$$\begin{aligned}\sum_{k=1}^{m} a(i,k)y(k) &\leq c(i)\frac{2\log\left(1 + \frac{na_i(\max)}{a_i(\min)}\right)}{B} \\ &= c(i)O\left(\frac{\log n + \log \frac{a_i(\max)}{a_i(\min)}}{B}\right).\end{aligned}$$

The proof of the theorem then follows directly from weak duality.

Proof of (1): Note first that when the value of $a_i(\max)$ increases, the value of the primal solution does not change. Thus, the value of the primal solution only increases when the dual solution increases. Initially, the values of the primal and dual solutions are zero. Consider the jth round in which $y(j)$ is being increased continuously. We prove that $\partial X(j)/\partial y(j) \leq B(\partial Y(j)/\partial y(j))$, concluding that $X(j) \leq B \cdot Y(j)$.

$$\begin{aligned}\frac{\partial X(j)}{\partial y(j)} &= \sum_{i=1}^{n} c(i)\frac{\partial x(i)}{\partial y(j)} \\ &\leq \sum_{i=1}^{n} \frac{c(i)Ba(i,j)}{2c(i)} \frac{1}{na_i(\max)} \exp\left(\frac{B}{2c(i)}\sum_{k=1}^{j} a(i,k)y(k)\right) \quad (14.1)\end{aligned}$$

$$
\begin{aligned}
&= \frac{B}{2}\sum_{i=1}^{n} a(i,j)\left[\frac{1}{na_i(\max)}\left(\exp\left(\frac{B}{2c(i)}\sum_{k=1}^{j} a(i,k)y(k)\right)-1\right)\right. \\
&\quad \left.+\frac{1}{na_i(\max)}\right] \\
&\leq \frac{B}{2}\sum_{i=1}^{n} a(i,j)\left[x(i)+\frac{1}{na_i(\max)}\right] \leq \frac{B}{2}(1+1) \\
&= B = B\frac{\partial Y(j)}{\partial y(j)}, \qquad (14.2)
\end{aligned}
$$

where inequality (14.1) follows by substituting value of the derivative of $x(i)$, and (14.2) follows since: (i) $\sum_{i=1}^{n} a(i,j)x(i) < 1$, (ii) $x(i) \geq (1/na_i(\max)[\exp((B/2c(i))\sum_{k=1}^{j} a(i,k)y(k)) - 1]$, and (iii) $(1/n)\sum_{i=1}^{n} a(i,j)/(a_i(\max)) \leq 1$. The final equality follows since the value of the dual is $\sum_{k=1}^{j} y(k)$, and so $\partial Y(j)/\partial y(j) = 1$.

Proof of (2): This claim is trivial since we increase the primal variables until the current primal constraint becomes feasible. We never decrease any $x(i)$, so (feasible) constraints remain feasible.

Proof of (3): Consider the ith dual constraint of the form $\sum_{k=1}^{j} a(i,k)y(k) \leq c(i)$. Each time a variable $y(k)$ with coefficient $a(i,k) > 0$ is increased, the primal variable $x(i)$ is increased too. Let $a_i(\min) = \min_{k=1}^{m}\{a(i,k)|a(i,k) \neq 0\}$ and $a_i(\max) = \max_{k=1}^{m}\{a(i,k)\}$ be as previously defined. During the run of the algorithm, $x(i) \leq 1/a_i(\min)$, since if equality holds, then each primal constraint $(1 \leq j \leq m)$ with $a(i,j) > 0$ is already feasible. Thus, we get the following:

$$
\frac{1}{na_i(\max)}\left[\exp\left(\frac{B}{2c(i)}\sum_{k=1}^{j} a(i,k)y(k)\right)-1\right] \leq x(i) \leq \frac{1}{a_i}(\min).
$$

Simplifying, we obtain:

$$
\sum_{k=1}^{j} a(i,k)y(k) \leq \frac{2\log\left(1+\frac{na_i(\max)}{a_i(\min)}\right)}{B}c(i).
$$

□

14.1.1 Lower Bounds

In this section, we prove a simple lower bound showing that the additional additive factor of $\log(a_i(\max)/a_i(\min))$ is indeed necessary.

Lemma 14.2. There is an instance of the general fractional packing problem with a single constraint such that

$$\sum_{j=1}^{m} a(i,j)y(j) \geq \frac{H(a(\max)/a(\min))}{B}$$

for any online B-competitive algorithm, where $H(m)$ denotes the mth harmonic number, and $a(\max)/a(\min)$ is the ratio between the maximum and minimum entries in the (single) constraint.

Proof. Consider the following instance, for any m:

$$\max \sum_{j=1}^{m} y(j)$$

$$\text{subject to} \sum_{j=1}^{m} (m-j+1)y(j) \leq 1 \ .$$

Note that for this instance $a(\max)/a(\min) = m$. The variables $y(j)$ arrive one-by-one. After the jth round (for each j), the optimal offline value is $1/(m-j+1)$. Thus, the value of the objective function given by a B-competitive algorithm must be at least $1/B(m-j+1)$. This yields the following sequence of inequalities:

$$\begin{aligned}
y(1) &\geq 1/(Bm), \\
y(1)+y(2) &\geq 1/(B(m-1)), \\
y(1)+y(2)+y(3) &\geq 1/(B(m-2)), \\
&\vdots \ \vdots \ \vdots \\
y(1)+y(2)+y(3)+\cdots+y(m) &\geq 1/B.
\end{aligned}$$

Summing up over all m inequalities we get the desired bound:

$$\sum_{j=1}^{m}(m-j+1)y(j) \geq \frac{1}{B}\sum_{j=1}^{m}\frac{1}{m-j+1} = \frac{H(m)}{B}.$$

□

14.2 The General Online Fractional Covering Problem

In this section, we describe our online scheme for computing a near-optimal fractional solution for the online fractional covering problem. Our scheme for the general online fractional covering problem gets a parameter $B > 0$. The competitive ratio of the scheme is $O(\log n/B)$ and for each constraint $\sum_{i=1}^{n} a(i,j)x(i) \geq 1/B$ holds. The scheme works in phases; when the first constraint is introduced, the scheme generates the first lower bound:

$$\alpha(1) \leftarrow \frac{1}{B}\min_{i=1}^{n}\left\{\frac{c(i)}{a(i,1)}\right\} \leq \frac{\mathrm{OPT}}{B}.$$

During the rth phase, it is assumed that the lower bound on the optimum is $\alpha(r)$, as long as the total primal cost does not exceed $\alpha(r)$. When the primal cost exceeds this bound, the scheme "forgets" about all the values given to the primal and dual variables so far, and starts a new phase in which the lower bound is doubled, i.e., $\alpha(r+1) \leftarrow 2\alpha(r)$. Nevertheless, the values of the "forgotten" variables are accounted for in the total cost of the solution. That is, the algorithm maintains in each phase r a new set of variables $x(i,r)$. However, since the variables of the linear program are required to be monotonically non-decreasing, the value of each variable $x(i)$ is actually set to $\max_r\{x(i,r)\}$ (or alternatively $\sum_r x(i,r)$). The cost of maintaining the variables of the linear program is, thus, at most the cost of maintaining the new variables in each phase. When we start processing a new phase we also set to zero all dual variables, and start processing again all primal constraints starting from the first one. Thus, in each such phase, our algorithm produces "fresh" primal and dual solutions.

In the following, we describe our scheme in a single round of the rth phase. Let $\sum_{i=1}^{n} a(i,j)x(i) \geq 1$ be the new primal constraint which is

currently introduced and let $y(j)$ be the corresponding dual variable. The values of the primal and dual variables are increased as follows. Note that during each phase $x(i)$ only increases. The performance of the scheme is analyzed in Theorem 14.3.

(1) $y(j) \leftarrow 0$
(2) While $\sum_{i=1}^{n} a(i,j)x(i) < 1/B$:
  (a) Increase $y(j)$ continuously.
  (b) Increase each variable $x(i)$ by the following increment function:

$$x(i) \leftarrow \frac{\alpha(r)}{2nc(i)} \exp\left(\frac{\log 2n}{c(i)} \sum_{k=1}^{j} a(i,k)y(k)\right).$$

Theorem 14.3. For any $B > 0$, the scheme for the general online fractional covering problem achieves a competitive ratio of $O(\log n/B)$, such that for each constraint $\sum_{i=1}^{n} a(i,j)x(i) \geq 1/B$.

Proof. Let $X(r)$ and $Y(r)$ be the values of the primal and dual solutions, respectively, generated during the rth phase. We prove the following claims on $X(r)$ and $Y(r)$:

(1) For each *finished* phase r: $Y(r) \geq B\alpha(r)/(2\log 2n)$.
(2) The dual solution generated during the rth phase is feasible.
(3) The total cost of the primal solutions generated from the first phase until the rth phase is less than $2\alpha(r)$.
(4) For any primal constraint given to the algorithm, $\sum_{i=1}^{n} a(i,j)x(i) \geq 1/B$.

From the first three claims together with weak duality we conclude that the total cost of the primal solutions in all the phases up to phase r is at most:

$$2\alpha(r) \leq 4\alpha(r-1) \leq 8\frac{\log 2n}{B} Y(r-1) \leq 8\frac{\log 2n}{B}\mathrm{OPT}.$$

Notice that if the scheme finishes in the first phase, then the total cost is at most $\alpha(1) \leq \mathrm{OPT}/B$.

Proof of (1): Initially, $x(i) = \alpha(r)/(2nc(i))$, and so $X(r)$ is initially at most $\alpha(r)/2$. The total profit of the dual solution is initially zero. From then on, the primal cost increases only when some dual variable $y(j)$ is increased. When the phase ends, $X(r) \geq \alpha(r)$. Thus, it suffices to prove that during the phase

$$\frac{\partial X(r)}{\partial y(j)} \leq \frac{\log 2n}{B} \frac{\partial Y(r)}{\partial y(j)}.$$

This follows since,

$$\begin{aligned}
\frac{\partial X(r)}{\partial y(j)} &= \sum_{i=1}^{n} c(i) \frac{\partial x(i)}{\partial y(j)} \\
&= \sum_{i=1}^{n} \frac{c(i) \log(2n) a(i,j)}{c(i)} \frac{\alpha(r)}{2nc(i)} \exp\left(\frac{\log 2n}{c(i)} \sum_{k=1}^{j} a(i,k) y(k) \right) \\
&= \log 2n \sum_{i=1}^{n} a(i,j) x(i) \leq \frac{\log 2n}{B} = \frac{\log 2n}{B} \frac{\partial Y(j)}{\partial y(j)}, \qquad (14.3)
\end{aligned}$$

where (14.3) follows since $\sum_{i=1}^{n} a(i,j)x(i) \leq 1/B$. The final equality follows since the value of the dual solution is $\sum_{k=1}^{j} y(k)$ and thus $\partial Y(j)/\partial y(j) = 1$.

Proof of (2): Consider the ith dual constraint of the form $\sum_{k=1}^{m} a(i,k)y(k) \leq c(i)$. Each time variable $y(k)$ with coefficient $a(i,k) > 0$ is increased, the corresponding primal variable $x(i)$ is increased too. During the rth phase of the algorithm, $x(i) \leq \alpha(r)/c(i)$, since otherwise it would have contributed to the cost of the primal solution more than $\alpha(r)$, and the current phase would have ended. Thus, we get the following equation:

$$x(i) = \frac{\alpha(r)}{2nc(i)} \exp\left(\frac{\log 2n}{c(i)} \sum_{k=1}^{j} a(i,k) y(k) \right) \leq \frac{\alpha(r)}{c(i)}.$$

Simplifying we get the desired result:

$$\sum_{k=1}^{m} a(i,k) y(k) \leq c(i).$$

Proof of (3): We bound the total cost paid by the online algorithm. The total primal cost in the rth phase is at most $\alpha(r)$. Since the ratio between $\alpha(k)$ and $\alpha(k-1)$ is 2, we get that the total cost until the rth phase is at most $\sum_{k=1}^{r} \alpha(k) \leq 2\alpha(r)$.

Proof of (4): The claim is trivial, since each round terminates only when the value of the left-hand side of the new primal constraint is at least $1/B$. The value of each variable $x(i)$ never decreases, thus all previous primal constraints remain feasible. □

14.3 Notes

The results in this chapter are based on the work of Buchbinder and Naor [32]. In this work, a more general setting is considered in which each variable in the covering problem also has an upper bound (box constraint).

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Conclusions and Further Research

We have shown in this survey how to extend the primal–dual method to the setting of online algorithms, establishing it as an important unifying methodology for the design of online algorithms, applicable to a wide variety of problems. These include, among others, the classic ski rental problem, the online set-cover problem, graph optimization problems, the dynamic TCP-acknowledgement problem, various routing problems, load balancing, machine scheduling, ad auctions, caching, and more. All these problems can now be solved using the basic recipe developed for online packing–covering problems, or simple tweaks thereof. It is important to note that it is rare to find general algorithmic ideas that work well for a wide variety of problems and we are certain that the online primal–dual framework will turn out to be useful for many more problems and algorithms.

Many open questions and directions for further research remain. First, it will be interesting to come up with other online scenarios that can benefit from the primal–dual framework. The k-server problem, the "holy grail" problem in competitive analysis, is a prime example. The online primal–dual framework seems very promising for resolving whether randomized algorithms can yield improved bounds for

the k-server problem. Another interesting problem is the metrical task problem in general metrics. It will be nice to use the primal–dual framework to close the gap between the upper and lower bounds. Another research direction is to extend the primal–dual framework itself beyond packing–covering formulations. We note that the framework cannot be extended to general linear programming formulations, however, there might be other special formulations that can be solved online.

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