Streaming Algorithms

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Outline

- Model
- Counting
- Heavy Hitters
- CountMin
- CountSketch
- Conclusion

Model

- Input: sequence of integers $x_1, x_2, \ldots, x_n$
  - $x_i \in U$
- Goal: compute some function f_i on the input stream

Focus

- Main focus:
  - Reduce the amount of memory used to process the input

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Counting

- Input: Sequence of integers $x_1, x_2, \ldots, x_n$
  - Problem: Find the number of elements in the input stream

Naïve Solution

- Keep a counter
  - Space Complexity: $O(\log_2 n)$ bits

Approx. Solution

- We would like to find an approximate solution $\hat{n}$ such that
  - Given $\epsilon > 0$ and $\delta < 1$
    - Find an estimate $\hat{n}$ such that:
      - $P(|n - \hat{n}| > \epsilon n) < \delta$

Morris' Algorithm

- Algorithm:
  - $X := 0$
  - For each item seen:
    - Increment X with probability $\frac{1}{2^X}$
  - Output: $\hat{n} := 2^{\mathrm{X}} - 1$
- Intuitively: $X \sim \log_2 n$

Analysis

- Let X_n denote the value of X after seeing the i^{th} item
  - We will show that:
    - $E(2^{X_n}) = n + 1$

Analysis

- Show that $E(2^{X_n}) = n + 1$
- Proof:
  - By induction on n
  - Base Case: $X_0 = 0$ => $E(2^{X_0}) = E(2^0) = 1$
  - Induction Hypothesis: Assume the claim is true for n prove for $n + 1$

Analysis

- Inductive Step:
  - $E(2^{X_{n+1}}) = \sum_{\forall i} 2^i\, P(X_{n+1} = i)$
  - $E(2^{X_{n+1}}) = \sum_{\forall i} 2^i\, (\, P(X_n = i-1) \cdot P(X_n\ gets\ incremented \mid X_n = i-1) + P(X_n = i) \cdot P(X_n\ does\ not\ gets\ incremented \mid X_n = i))$
  - $E(2^{X_{n+1}}) = \sum_{\forall i} 2^i \left(P(X_n = i-1) \cdot \frac{1}{2^{i-1}} + P(X_n = i) \cdot \left(1 - \frac{1}{2^i}\right)\right)$
  - $E(2^{X_{n+1}}) = \sum_{\forall i} 2^i\ P(X_n = i-1) \cdot \frac{1}{2^{i-1}} + \sum_{\forall i} 2^i\, P(X_n = i) \cdot \left(1 - \frac{1}{2^i}\right)$
  - $E(2^{X_{n+1}}) = \sum_{\forall i} 2^i\, P(X_n = i-1) \cdot \frac{1}{2^{i-1}} + \sum_{\forall i} 2^i\, P(X_n = i)\ - \sum_{\forall i} 2^i\, P(X_n = i) \cdot \frac{1}{2^i}$
  - $E(2^{X_{n+1}}) = \sum_{\forall i} 2\, P(X_n = i-1) + \sum_{\forall i} 2^i\, P(X_n = i)\ - \sum_{\forall i} P(X_n = i)$
  - $E(2^{X_{n+1}}) = \sum_{\forall i} 2^i\, P(X_n = i) + \sum_{\forall i} P(X_n = i)$
  - $E(2^{X_{n+1}}) = E(2^{X_n}) + 1 = (n+1) + 1$

Analysis

- *We can show the following:*
  - $E(2^{X_n}) = n + 1$
  - $E(2^{2X_n}) = \frac{3}{2}n^2 + \frac{3}{2}n + 1$ (similar to previous proof)
- Therefore, we have that:
  - $Var(2^{X_n}) < \frac{1}{2}n^2$
- Hence: By Chebyshev's inequality
  - $P(|n - \hat{n}| > \epsilon n) < \frac{1}{2\epsilon^2}$
- Note: This is not very useful when $\epsilon \geq 1$
  - We will improve this algorithm soon

Space Complexity

- *Since we have:* $E(2^{X_n}) = n + 1$.
  - $We\ can\ show:\ P(2^{X_n} - 1 \geq n^c) \leq \frac{1}{n^{c-1}}$
- *Meaning, with high probability we have*
  - $2^{X_n} - 1 \geq n^c$
  - $2^{X_n} \geq n^c - 1$
  - $X_n \geq \log_2(n^c - 1)$
- *Hence, to store* X_n *we can show that we need*
  - $\mathrm{O}(\log_2(\log_2 n))$ *bits with high probability*

Morris+ Algorithm

- Improve Morris' algorithm by using the mean trick
  - *Run* $s > 1$ *independent copies of Morris' algorithm and average their outputs*

Morris+ Algorithm

- *Let X^j be the output of the j^{th} copy of Morris' algorithm* after seeing the i^{th} item
  - $Y_i = \frac{1}{s}\sum_j \left(2^{X^j} - 1\right)$
  - By linearity of expectation we have $E(2^{Y_n}) = n + 1$
  - But
    - $Var(2^{Y_n}) < \frac{1}{2s}n^2 < Var\left(2^{X_n^j}\right) = \frac{1}{2}n^2$
- By Chebyshev's inequality we have:
  - $P(|n - \hat{n}| > \epsilon n) < \frac{1}{2s\epsilon^2}$
  - Then for δ error probability we set
    - $s > \frac{1}{2\epsilon^2\delta}$

Morris+ Space

- Space complexity: $O(s \cdot log_2(log_2 n))$ bits with high probability
- *For* δ error probability *we need*
  - $O(\frac{1}{2\epsilon^2\delta} \cdot log_2(log_2 n))$ bits with high probability
- We will improve this space complexity using the median trick

Morris++ Algorithm

- Improve space complexity by using the median trick
  - *Run t independent copies of Morris+ algorithm*
    - Such that $s = \frac{3}{2 \cdot \epsilon^2}$
      - Meaning the error probability of each Morris+ is $\frac{1}{3}$
    - Output the median estimate

Morris++ Algorithm

- Note:
  - Since the error probability of each Morris+ is $\frac{1}{3}$
    - Expected number of Morris+ instantiations that succeed is $\frac{2t}{3}$
  - Hence, for the median to be a bad estimate at least half of the Morris+ instantiations must fail
  - We will show that this is not likely

Morris++ Algorithm

- Let $Z_i = 1$ if i^{th} Morris+ instantiation succeeds, otherwise $Z_i = 0$
  - We will bound $P\left(\sum_i Z_i \le \frac{t}{2}\right)$
    - $P\left(\sum_i Z_i \le \frac{t}{2}\right) \le P\left(\left|\sum_i Z_i - \frac{2t}{3}\right| \le \frac{t}{2} - \frac{2t}{3}\right)$
    - $P\left(\sum_i Z_i \le \frac{t}{2}\right) \le P\left(\left|\sum_i Z_i - E(\sum_i Z_i)\right| \le -\frac{t}{6}\right)$
  - By Hoeffding bound
    - $P\left(\sum_i Z_i \le \frac{t}{2}\right) \le e^{\frac{-1}{18}t}$
    - Hence, for $t \ge \left\lceil 18 \ln \frac{1}{\delta} \right\rceil$
      - $P\left(\sum_i Z_i \le \frac{t}{2}\right) \le \delta$

Morris++ Space

- If we set $s \cdot t = \theta\left(\frac{1}{\epsilon^2}\ln\frac{1}{\delta}\right)$
  - *We get that we need* $O\left(\frac{1}{\epsilon^2}\ln\left(\frac{1}{\delta}\right) \cdot \log_2\left(\log_2 n\right)\right)$ bits with high probability

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Heavy Hitters

- Input: sequence of integers $x_1, x_2, \dots, x_n$
  - $x_i \in \{1, 2, \dots, m\}$
- Let f_i be the frequency of element i in the given sequence
- Goal: Given some integer K find the elements that have $f_i > \frac{n}{K}$
- There is a simple two pass algorithm named Misrea – Gries Algorithm (was covered last time)

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CountMin Algorithm

- Pick t hash functions such that $h_i: [m] \to [w]$ from a universal family of hash functions
- Create a 2D array $C[t][w]$ initially all cells set to 0
- Algorithm:
  - For each item x:
    - For i from 1 to t
      - Increment $C[i][h_i(x)]$
- Then the frequency of item x is $\min_{\forall i} C[i][h_i(x)]$

CountMin Algorithm

Fun	$C[t][0]$	$C[t][1]$	$C[t][2]$
h_0	0	0	0
h_1	0	0	0
h_2	0	0	0

Insert 3

3

Fun	$C[t][0]$	$C[t][1]$	$C[t][2]$
h_0	0	0	0
h_1	0	0	0
h_2	0	0	0

Insert 3

Fun	$C[t][0]$	$C[t][1]$	$C[t][2]$
h_0	1	0	0
h_1	0	1	0
h_2	0	0	1

Insert 5

5

Fun	$C[t][0]$	$C[t][1]$	$C[t][2]$
h_0	1	0	0
h_1	0	1	0
h_2	0	0	1

Insert 5

Fun	$C[t][0]$	$C[t][1]$	$C[t][2]$
h_0	1	1	0
h_1	0	2	0
h_2	1	0	1

Insert 10

10

Fun	$C[t][0]$	$C[t][1]$	$C[t][2]$
h_0	1	1	0
h_1	0	2	0
h_2	1	0	1

Insert 10

Fun	$C[t][0]$	$C[t][1]$	$C[t][2]$
h_0	1	1	1
h_1	0	3	0
h_2	2	0	1

Query 3

Minimum

Fun	$C[t][0]$	$C[t][1]$	$C[t][2]$
h_0	1	1	1
h_1	0	3	0
h_2	2	0	1

Query 5

Minimum

Fun	$C[t][0]$	$C[t][1]$	$C[t][2]$
h_0	1	1	1
h_1	0	3	0
h_2	2	0	1

CountMin Algorithm

- We will do some analysis on this algorithm
- Let f_x be the actual count of x
- Let $\widehat{f_x}$ be the estimated count of x
- Note: $\mathrm{f_x} \leq \widehat{f_x}$
- We will show that $\mathrm{P}\left(\widehat{f_x} \geq f_x + \epsilon n\right) \leq \delta$

CountMin Algorithm

- We will compute the $\mathrm{E}\left(C[j][h_j(x)]\right)$
  - $\mathrm{E}\left(C[j][h_j(x)]\right) = \mathrm{E}\left(\sum_{\forall s: h_j(s)=h_j(x)} f_s\right)$
  - $\mathrm{E}\left(C[j][h_j(x)]\right) = f_x + \frac{1}{w}\sum_{\forall s \neq x} f_s$
  - $\mathrm{E}\left(C[j][h_j(x)]\right) < f_x + \frac{n}{w}$

CountMin Algorithm

- We have
  - $\mathrm{E}\big(C[j][h_j(x)]\big) < f_x + \frac{n}{w}$
- We will bound $\mathrm{P}\left(C[j][h_j(x)] \geq f_x + \frac{2n}{w}\right)$
  - $\mathrm{P}\left(C[j][h_j(x)] \geq f_x + \frac{2n}{w}\right) \leq \mathrm{P}\left(C[j][h_j(x)] - f_x \geq \frac{2n}{w}\right)$
  - By Chebyshev's inequality:
  - $\mathrm{P}\left(C[j][h_j(x)] \geq f_x + \frac{2n}{w}\right) \leq \frac{E\left(C[j][h_j(x)] - f_x\right)}{\frac{2n}{w}}$
  - $\mathrm{P}\left(C[j][h_j(x)] \geq f_x + \frac{2n}{w}\right) \leq \frac{f_x + \frac{n}{w} - f_x}{\frac{2n}{w}} \leq \frac{1}{2}$

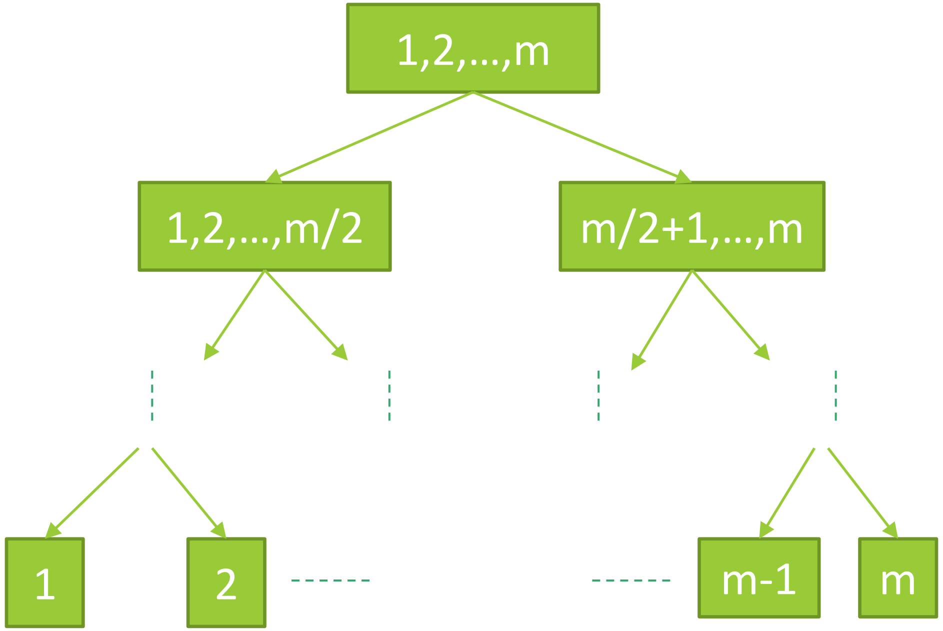
CountMin Algorithm

- So far we have:
  - $\mathrm{P}\left(C[j][h_j(x)] \geq f_x + \frac{2n}{w}\right) \leq \frac{1}{2}$
- We will bound $\mathrm{P}\left(\widehat{f_x} \geq f_x + \frac{2n}{w}\right)$
  - $\mathrm{P}\left(\widehat{f_x} \geq f_x + \frac{2n}{w}\right) = \mathrm{P}\left(\min_{\forall \mathrm{j}} C[j][h_j(x)] \geq f_x + \frac{2n}{w}\right)$
  - $\mathrm{P}\left(\widehat{f_x} \geq f_x + \frac{2n}{w}\right) = \prod_{\mathrm{j}} \mathrm{P}\left(C[j][h_j(x)] \geq f_x + \frac{2n}{w}\right)$
  - $\mathrm{P}\left(\widehat{f_x} \geq f_x + \frac{2n}{w}\right) \leq \left(\frac{1}{2}\right)^t$
  - If we set w$= \frac{2}{\epsilon}$ and $\mathrm{t} = \log_2 \frac{1}{\delta}$ we will have
    - $\mathrm{P}\left(\widehat{f_x} \geq f_x + \epsilon n\right) \leq \delta$

CountMin Algorithm

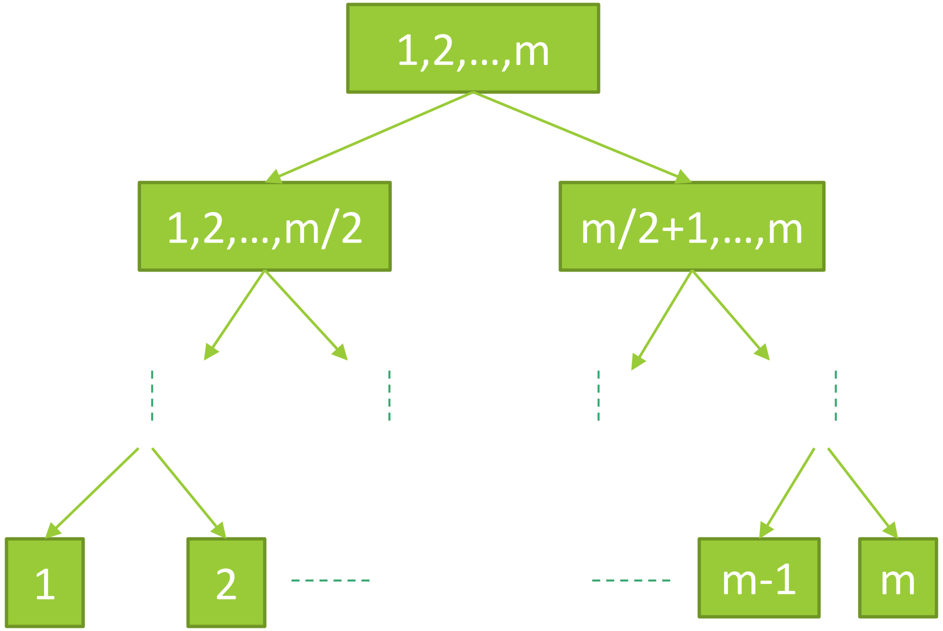
- Space complexity:
  - $\mathrm{O}(\mathrm{w} \cdot t) = O\left(\frac{2}{\epsilon} \cdot \log_2 \frac{1}{\delta}\right)$

Heavy Hitters with CountMin



- We extend CountMin as follows:
  - For each row of intervals in figure, we store a separate count-min structure
  - For each row, count-min of that row treats two elements that fall into the same interval as the same element
  - Note that the value at any ancestor of a node is at least as big as the value at that node

Heavy Hitters with CountMin



- To get the K heavy-hitters:
  - Explore the tree starting from the root
    - Only explore the children of intervals that have frequency at least $\frac{n}{K}$

Heavy Hitters with CountMin

Analysis:

- Space complexity $O\left(\frac{2}{\epsilon} \cdot \log_2 \frac{1}{\delta} \cdot \log_2 n\right)$
- Time complexity to get K heavy hitters is $O(K \cdot \log_2 n)$
  - For any given row, the sum over all frequencies in that row is n
  - Thus, in any row, there are at most K intervals with frequency $\frac{n}{K}$
  - Therefore, we only explore the children of at most K intervals in any given row

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CountSketch Algorithm

- Pick t hash functions such that $h_i: [m] \to [w]$ from a universal family of hash functions
- Pick t hash functions such that $s_i: [m] \to \{-1, +1\}$ from a universal family of hash functions
- Create a 2D array $C[t][w]$ initially all cells set to 0
- Algorithm:
  - For each item x:
    - For i from 1 to t
      - $C[i][h_i(x)] = C[i][h_i(x)] + s_i(x)$
- Then the frequency of item x is $\widehat{f_x} = \underset{\forall\, i}{\mathrm{median}} \{C[i][h_i(x)] \cdot s_i(x)\}$

CountSketch Algorithm

- We can show that
  - When we set $t = O(\log n)$ and $\mathrm{w} = \frac{3}{\epsilon^2}$
  - Then, with high probability
    - $\left|\widehat{f_x} - f_x\right| \leq \epsilon \cdot \left(\sum_j f_j^2\right)$
    - $\left(\sum_j {f_j}^2\right) \ll n$ for skewed distributions

CountSketch Algorithm

- Space complexity:
  - $O\left(\frac{1}{\epsilon^2} \cdot \log_2 \frac{1}{\delta}\right)$

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Conclusion

- Randomized approximate algorithms provide simple solutions to important problems
- Mean and median tricks help us improve the error probability and space complexity of algorithm

References

- The material presented is from the following source:
  - https://www.sketchingbigdata.org/fall20/lec/notes.pdf
- I have used the following resources to understand some of the proofs better:
  - http://www.cs.columbia.edu/~andoni/s17_advanced/algorithms/mainSpace/files/scribe1.pdf
  - http://www.cs.columbia.edu/~andoni/s17_advanced/algorithms/mainSpace/files/scribe2.pdf
  - http://www.cs.columbia.edu/~andoni/s17_advanced/algorithms/mainSpace/files/scribe5.pdf
  - http://web.stanford.edu/class/cs369g/files/lectures/lec7.pdf
  - http://web.stanford.edu/class/cs369g/files/lectures/lec8.pdf