Online Graph Coloring

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Online Graph coloring

Input sequence: $(v_1, N_1), \ldots, (v_n, N_n)$ where $v_i \prec v_{i+1}$ and $N_i = N(v_i) \cap \{v_j : j < i\}$

Output: $c : V \to [k]$

Goal: Minimize k. k is the number of color used.

Chromatic number: Smallest number of need for coloring. Denoted as χ .

Lower bound

Theorem: For every **deterministic online algorithm** there exists a logn-colorable graph for which the algorithm uses at least 2n/logn colors. The performance ratio of any deterministic online coloring algorithm is at least $\frac{2n}{\log^2 n}$.

Transparent online coloring game

A poses a pair $(v_t, \mathrm{Adj}^-(v_t))$ where $\mathrm{Adj}^-(v_t) \subseteq \{v_1, \ldots, v_{t-1}\}$.
B answers with an integer $\mathrm{Bin}(v_t)$, a proper coloring of v_t.
A responds with an integer $\mathrm{Col}(v_t)$, a proper coloring of v_t.

Adversary strategy

$[k]^{k/2}$: The collection of all subsets of {1,2,...,k} of size k/2.

Avail(vt): Admissible colors consists of colors not used by its pre-neighbors.

Hue(b)={Corlor(vi): Bin(vi) = b}: hue of a bin is the set of colors of vertices in the bin.

H: hue collection is a set of all nonempty hues.

$$n = \frac{k}{2}\binom{k}{k/2},\ \textit{while } t \le n: \quad \begin{array}{c} \mathrm{Avail}(v_t) = \mathrm{any}([k]^{(k/2)} - H) \\ \mathrm{Adj}^-(v_t) = \{v_i\colon \mathrm{Col}(v_i) \notin \mathrm{Avail}(v_t) \text{ and } i < t\} \\ \mathrm{Col}(v_t) = \mathrm{any}(\mathrm{Avail}(v_t) - \mathrm{Hue}(\mathrm{Bin}(v_t))) \end{array}$$

#bin >= n/(k/2)
#color<=k
ratio>=2n/(k*k)

Lower bound

Theorem: For every **randomized online algorithm** there exists a k-colorable graph on which the algorithm uses at least n/k bins, where k=O(logn). The performance ratio of any randomized online coloring algorithm is at least $\frac{n}{16\log^2 n}$.

Adversary strategy for randomized algo

$$\text{Avail}(v_t) = \text{Random}([k]^{(k/2)})$$
$$\text{Adj}^-(v_t) = \{v_i\colon \text{Col}(v_i) \notin \text{Avail}(v_t) \text{ and } i < t\}$$
$$\text{Col}(v_t) = \text{Random}(\text{Avail}(v_t))$$

Relaxing the constraint - blocked input

Theorem: The performance ratio of any randomized algorithm, when the input is presented in blocks of size $O\left(\log^2 n\right)$, is $\Omega\left(\frac{n}{\log^2 n}\right)$.

Relaxing other constraints

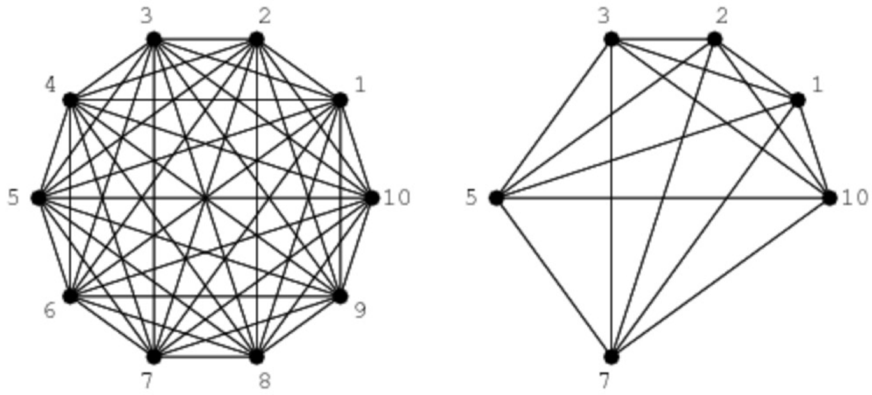
1. Look-ahead and bufferring
2. Recoloring
3. Presorting vertices by degree
4. Disclosing the adversary’s previous coloring

First Fit

Use the smallest numbered color that does not violate the coloring requirement

Induced subgraph

A **induced subgraph** is a subset of the vertices of a graph G together with any edges whose endpoints are both in the subset.



Perfect graph

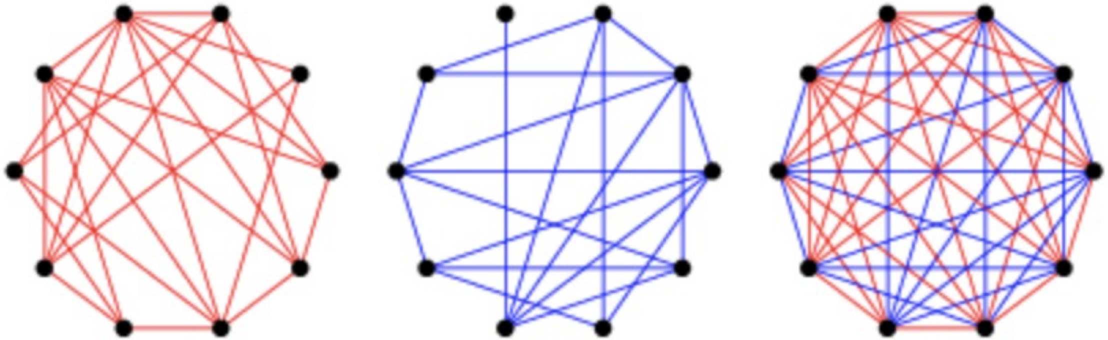
1. A **clique** of a graph *G* is an induced subgraph of *G* that is complete.

1. The **clique number** of a graph G, denoted omega(G), is the number of vertices in a maximum clique of G.

1. A **perfect graph** is a graph G such that for every induced subgraph of G, the clique number equals the chromatic number

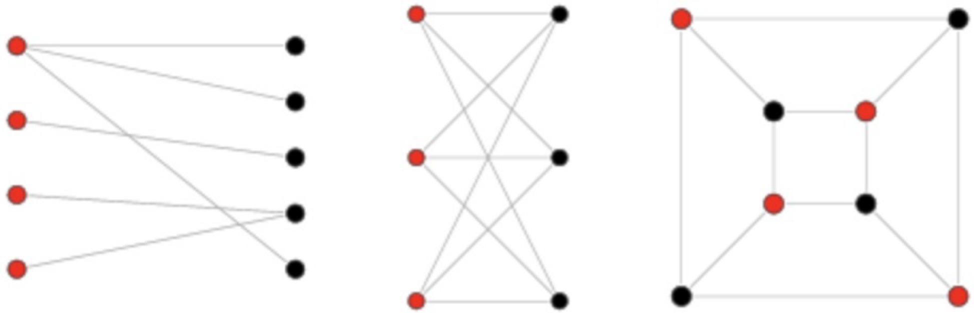
Complement graph

The complement of a graph G, is the graph G' with the same vertex set but whose edge set consists of the edges not present in G (i.e., the complement of the edge set of G with respect to all possible edges on the vertex set of G).



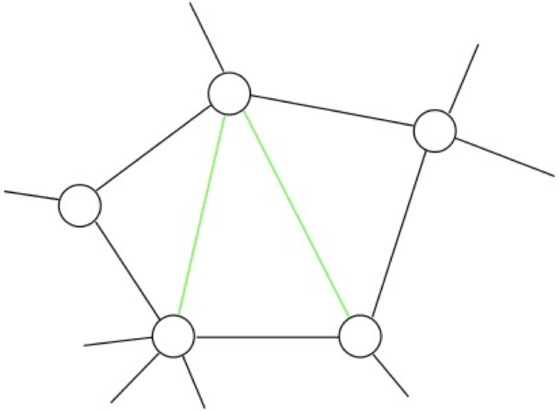
Bipartite graph

A bipartite graph is a set of graph vertices decomposed into two disjoint sets such that no two graph vertices within the same set are adjacent.



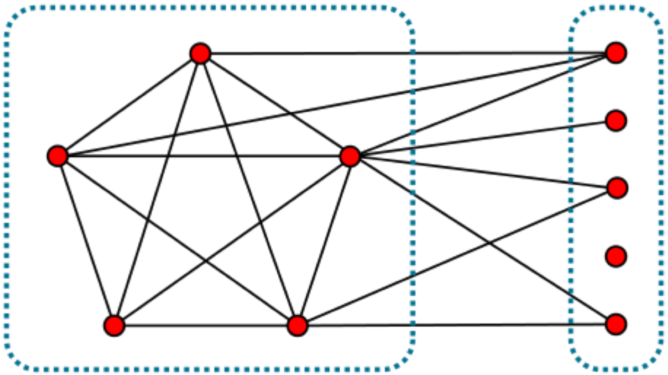
Chordal graph

A **chordal graph** is one in which all cycles of four or more vertices have a chord, which is an edge that is not part of the cycle but connects two vertices of the cycle. Equivalently, every induced cycle in the graph should have exactly three vertices.



Split graph

A **split graph** is a graph in which the vertices can be partitioned into a clique and an independent set.



FF for perfect graph

1. If G is a split graph:

$$\chi_{FF}(G) \leq \omega(G) + 1$$

1. If G is the complement of a bipartite graph:

$$\chi_{FF}(G) \leq \frac{3}{2}\omega(G)$$

1. If G is the complement of a chordal graph:

$$\chi_{FF}(G) \leq 2\omega(G) - 1$$

FF for interval graph

If G is an interval graph

$$\boldsymbol{\chi}_{FF}(G) \leq c\,\boldsymbol{\omega}(G)\log(\boldsymbol{\omega}(G))$$

Tree

Theorem 1: For every positive integer n there exists a tree Tn such that $\chi_A(Tn) \geq n$ holds for every online algorithm A.

Theorem 2: For each deterministic online algorithm ALG, ALG uses at least logn colors to color tree with n nodes. In other words, the competitive ratio >= logn/2.

Theorem 3: FirstFit coloring uses at most floor(logn)+1 colors to color any tree with n nodes. Therefore, FirstFit for trees has competitive ratio logn/2 .

Randomized Online Graph Coloring

Theorem: There exists a randomized online graph coloring algorithm A such that for every graph G of chromatic number χ , and every ordering on the vertices of G, the expected number of colors used by A is

$$O\left(\chi 2^{\chi} n^{(x-2)/(x-1)} (\log n)^{1/(x-1)}\right)$$

Definition: By a partial greedy s-coloring of G we mean that we greedily color G with s colors leaving out vertices that cannot be colored. We call the set of vertices left uncolored the **residue set**.

Randomized Online Graph Coloring

Theorem: For any randomized online coloring algorithm A there exists an n-vertex, χ - colorable graph G so that the expected number of colors A uses to color G is **at least**

$$\Omega\left(\frac{1}{\chi - 1}\left(\frac{\log n}{12(\chi + 1)}\right)^{\chi - 1}\right)$$