Online graph coloring

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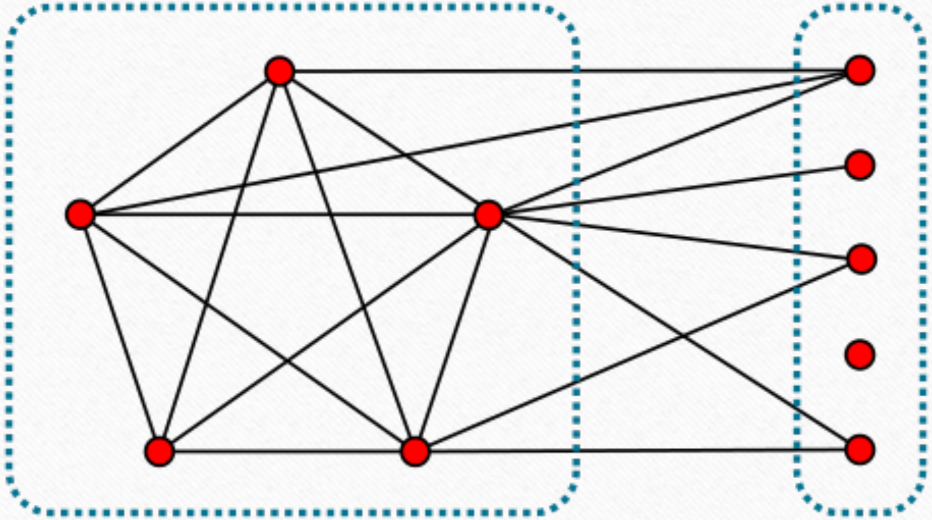
CSC2421-presentation2

First Fit

- Use the smallest numbered color that does not violate the coloring requirement.
- FF produces a **maximal stable** sequence partition $V(G) = S_1 \cup \cdots \cup S_k$, where S_i is a maximal nonempty stable set in the subgraph induced by $S_i \cup \cdots \cup S_k$
- $\chi_{FF}(G) \leq (2 + \varepsilon)\ \chi\ (G)$ holds for almost all graphs.

Upper Bound for Online First Fit Coloring

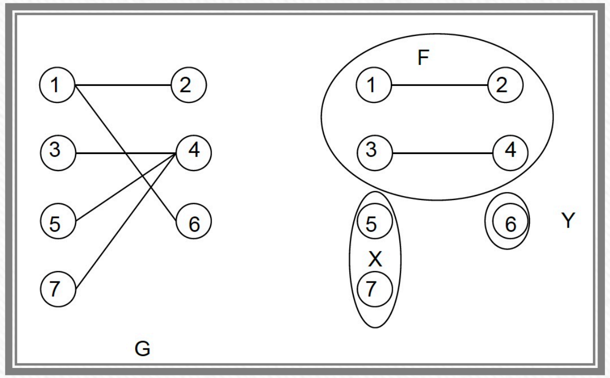
- **Theorem**: If G is a split graph(the union of a clique and a stable set with arbitrary edges between them), then $\chi_{FF}(G) \leq \omega(G) + 1$.



Upper Bound for Online First Fit Coloring

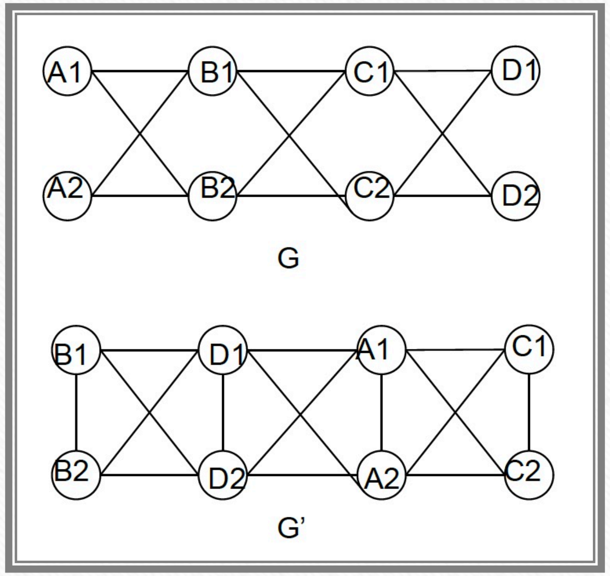
- **Theorem**: If G is a bipartite graph, $\bar{G}$ is the complement of G, then $\chi_{FF}(\bar{G}) \leq \frac{3}{2}\omega(\bar{G})$.
  - A maximal clique sequence partition of G consists of a set F of independent edges and a stable set $X \cup Y$ of the nonsaturated vertices(X and Y are, respectively in the first and second bipartition class of G).
  - $|\mathrm{F}| + |X| \leq \omega(\bar{G}), |\mathrm{F}| + |Y| \leq \omega(\bar{G})$,

    $|X \cup Y| = |X| + |Y| \leq \omega(\bar{G})$
  - $\chi_{FF}(\bar{G}) = |\mathrm{F}| + |X \cup Y| \leq \frac{3}{2}\omega(\bar{G})$,



Upper Bound for Online First Fit Coloring

- Why $\frac{3}{2}\omega(\bar{G})$ is a tight bound?
    - Let $V(G) = A \cup B \cup C \cup D$, where A, B, C and D are pairwise disjoint stable sets of k vertices.
    - $A \cup B, B \cup C$ and $C \cup D$ induced complete bipartite subgraph.
    - $\chi_{FF}(\bar{G})$= 3k and $\omega(\bar{G})$= 2k.



Upper Bound for Online First Fit Coloring

- **Theorem**: If G is a chordal graph, $\bar{G}$ is the complement of G, then $\chi_{FF}(\bar{G}) \leq 2\omega(\bar{G}) - 1$.
  - Let $C_1, \dots, C_k$ be a first fit clique partition of G. C_i is a maximal clique in the subgraph of G induced by $C_i \cup \dots \cup C_k$. Want to show $\omega\,(\bar{G}) \geq \frac{k+1}{2}$ by induction. Let $G' = G \backslash V(C_1)$. $C_2, \dots, C_k$ be a first fit clique partition of G'.
  - Case1: If G' has more components than G. Easily to use IH.
    - Proof from the presenter: Let $G' = G_1 \cup \cdots G_m$
    - $\omega(\overline{G'}) = \omega(\overline{G_1}) + \cdots + \omega(\overline{G_m}) \geq \frac{k_1+1}{2} + \cdots + \frac{k_m+1}{2}$ $where\ k_1 + \cdots + k_m = k - 1$.

$$= \frac{k_1+\cdots+k_m+m}{2} = \frac{\text{k}-1+m}{2} \geq \frac{\text{k}+1}{2}$$ as wanted.

  - Case2 : Otherwise. G has a simplicial vertex whose neighborhoods induce a clique in C1. This vertex can be added to any maximum independent set of G', and $\omega(\bar{G}) \geq \omega(\overline{G'}) + 1 \geq \frac{(\text{k}-1)+1}{2} + 1 > \frac{\text{k}+1}{2}$.

No Bounded Algo For Chordal

- **Theorem**: For every positive integer n there exists a tree T_n such that $\chi_A(T) \geq n$ holds for every online algorithm A.
  - Base case: n=1. Let T_1 be a single vertex tree.
  - n>1. Assume $T_1, \dots, T_{n-1}$ have been defined.
    - Make copies of $T_1, \dots, T_{n-1}$ and in all copies we distinguish distinct vertices as roots. T_n is formed as the union of all these rooted copies of $T_1, \dots, T_{n-1}$ plus a new vertex x joined to every root.

Deterministic Online Algo for Chordal

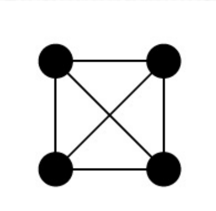
Theorem 1. *Let $d \in \mathbb{N}$ with $d \geq 2$ be arbitrary. For every deterministic online algorithm $\mathcal{A}$ and every $n \in \mathbb{N}$ with $n \geq 2d^2$, there exists a n-vertex chordal graph G with chromatic number $\chi(G) = d$ such that $\mathcal{A}$ uses $\Omega(d \cdot \log n)$ colors to color G.*

Lemma 1. *Let $d \in \mathbb{N}$ with $d \geq 2$ be arbitrary. For every deterministic online algorithm $\mathcal{A}$ and every $k \in \mathbb{N}$, there exists a chordal graph G_k having chromatic number $\chi(G_k) = d$ and consisting of $n_k \leq d2^k$ vertices such that $\mathcal{A}$ is forced to use at least $c_k \geq (d-1)k/4$ colors to color G_k.*

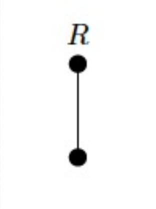
1. ADV constructs a chordal G_k recursively.
2. Gk has a forest representation. In every tree T of G_k, each tree node represents a clique of size d/2 in G_k. So, tree node != vertex.
3. If two tree nodes u_T and v_T are connected by a tree edge in T, then any two vertices $u \in u_T$ and $v \in v_T$ are connected by an edge in G_k . Hence u_T and v_T form a clique of size d in G_k .

Deterministic Online Algo for Chordal

- Notations:
  - Each tree T in G_k has a root node. Let $r(T)$ be the set of these $d/2$ vertices.
  - Let $r(G_k)$ be the union of $r(T)$.
  - For any subset V' of the vertices of G_k, let $C_A(V')$ be the set of colors used to color V'.
- Example:



Real G



Tree representation of G.

Deterministic Online Algo for Chordal

- For increasing k, the following invariants will be maintained.

(1) Algorithm $\mathcal{A}$ uses at least $\frac{d}{4} \cdot k$ colors for the root vertices of G_k, i.e. $|\mathcal{C}_{\mathcal{A}}(r(G_k))| \geq \frac{d}{4} \cdot k$.

(2) G_k is a union of connected components, each of which can be represented by a tree T. Each tree node is a clique of size $d/2$. Every tree T has a distinguished root node containing a set $r(T)$ of $d/2$ root vertices in G_k.

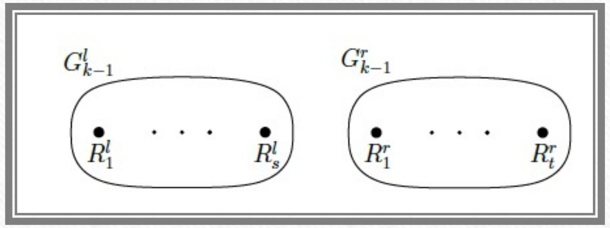
(3) G_k is chordal.

(4) The maximum clique size is $\omega(G_k) = d$.

(5) The number of vertices satisfies $n_k \leq \frac{d}{2} \cdot (2^{k+1} - 1)$.

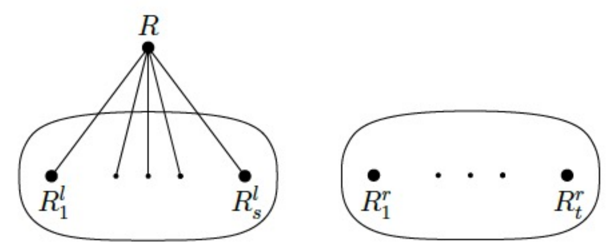
Deterministic Online Algo for Chordal

- Base graph G_1: Clique of size d. Only one tree. Pick any d/2 vertices be the root node and the remaining d/2 vertices be the second tree node.
- k>1:
  - Invariant (1) implies $\left|C_A\left(r\left(G_{k-1}^{l}\right)\right)\right| \geq (k-1)d/4$ and $\left|C_A\left(r\left(G_{k-1}^{r}\right)\right)\right| \geq (k-1)d/4$
  - Case1: If $\left|C_A\left(r\left(G_{k-1}^{l}\right) \cup r\left(G_{k-1}^{r}\right)\right)\right| \geq kd/4,\ G_k = \left(G_{k-1}^{l}\right) \cup \left(G_{k-1}^{r}\right)$. No further vertices added.



Deterministic Online Algo for Chordal

- k>1, Case2: If $\left|C_A\left(r(G_{k-1}^l) \cup r(G_{k-1}^r)\right)\right| < \frac{dk}{4}$. Add new root node R. For every vertex of R there is an edge to every vertex in R_i^l. There is a tree edge between R and every R_i^l.
  - Let $q = \left|C_A(r(G_{k-1}^r)) \setminus C_A\left(r(G_{k-1}^l)\right)\right| = \left|C_A\left(r(G_{k-1}^r) \cup r(G_{k-1}^l)\right)\right| - \left|C_A\left(r(G_{k-1}^l)\right)\right| < \frac{dk}{4} - \frac{d(k-1)}{4} = \frac{d}{4}$
  - $C_A(r(G_{k-1}^r)) = \left[C_A\left(r(G_{k-1}^l)\right) \cap C_A(r(G_{k-1}^r))\right] \cup \left[C_A(r(G_{k-1}^r)) \setminus C_A\left(r(G_{k-1}^l)\right)\right]$
  - R is a clique of size $\frac{d}{2}$, must use at least $\frac{d}{2} - q > \frac{d}{4}$ colors not in $C_A\left(r(G_{k-1}^l)\right)$.
  - $|C_A(r(G_k))| = |C_A(R \cup r(G_{k-1}^r))| \geq \frac{d}{4} + \frac{d(k-1)}{4} = \frac{dk}{4}$



- Can create a connected graph by adding a final vertex v_f that has an edge to exactly one root vertex in each of the components.
  - $\#vertices \leq \frac{d}{2}(2^{k+1} - 1) + 1 \leq d2^k$

Deterministic Online Algo for Chordal

Lemma 1. *Let $d \in \mathbb{N}$ with $d \geq 2$ be arbitrary. For every deterministic online algorithm $\mathcal{A}$ and every $k \in \mathbb{N}$, there exists a chordal graph G_k having chromatic number $\chi(G_k) = d$ and consisting of $n_k \leq d2^k$ vertices such that $\mathcal{A}$ is forced to use at least $c_k \geq (d-1)k/4$ colors to color G_k.*

(1) Algorithm $\mathcal{A}$ uses at least $\frac{d}{4} \cdot k$ colors for the root vertices of G_k, i.e. $|\mathcal{C}_{\mathcal{A}}(r(G_k))| \geq \frac{d}{4} \cdot k$.

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Deterministic Online Algo for Chordal

Theorem 1. *Let $d \in \mathbb{N}$ with $d \geq 2$ be arbitrary. For every deterministic online algorithm $\mathcal{A}$ and every $n \in \mathbb{N}$ with $n \geq 2d^2$, there exists a n-vertex chordal graph G with chromatic number $\chi(G) = d$ such that $\mathcal{A}$ uses $\Omega(d \cdot \log n)$ colors to color G.*

- Given d and n, let $k = \left\lfloor \log \frac{n}{d} \right\rfloor$, by lemma, there exists a chordal G_k with $n_k \leq d2^k$ vertices, chromatic number d and algo uses at least $c_k \geq \frac{(d-1)k}{4}$ colors.
- $n_k \leq d\frac{n}{d} \leq n$, add $n - n_k$ vertices, all of which have one edge to an arbitrary vertex of G_k.
- $d \leq \sqrt{n/2}$ since $n \geq 2d^2$. So $k \geq \log n - \log d - 1 \geq logn - \frac{1}{2}\log\frac{n}{2} - 1 = \log\frac{n}{2} - \frac{1}{2}\log\frac{n}{2} = \frac{1}{2}\log\frac{n}{2}$
- Since $n \geq 2d^2 \geq 4, \log\frac{n}{2} \geq \frac{1}{2}\log n$
- Since $d \geq 2,\ c_k \geq \frac{dk}{8} \geq \left(\frac{1}{2}\log\frac{n}{2}\right)\frac{d}{8} \geq \frac{d}{16}\left(\frac{1}{2}\log n\right) = \frac{d\log n}{32}$

Lower Bound for Deterministic Online Coloring

- **Theorem:** For every **deterministic online algorithm** there exists a logn-colorable graph for which the algorithm uses at least 2n/logn colors. The performance ratio of any deterministic online coloring algorithm is at least $\frac{2\text{n}}{\log^2 n}$.
  - $Avail(v_t)$=admissible colors consists of the colors not used by its pre-neighbors
  - $Hue(b) = \{Col(vi): Bin(vi) = b\}$
  - H is a hue collection set of all nonempty hues.
  - $[k]\ is\ the\ set\ \{1,2, \ldots, k\}$.
  - $[k]^{\left(\frac{k}{2}\right)}$ is the collection of all subsets of [k] of size k/2.
  - Algo colors with bins and the adversary with colors.

Lower Bound for Deterministic Online Coloring

- Observation1: $Hue(Bin(v_t)) \subseteq Avail(v_t\ )$
- Observation2: If m distinct bin/color pairs have been assigned after round t, then at least $m/(\max_{i\le t}|Avail(v_i)|)$ bins have been used.
- Observation3: If $Col(v_t) \in Avail(v_t) - Hue(Bins(v_t))$ then progress is made.

- Adversary Strategy:
  - Let $n = \frac{k}{2}\binom{k}{k/2}$, while $t \le n$:
  - $Avail(v_t) = any\left([k]^{k/2} - H\right)$
  - $Adj^{-}(v_t) = \{v_i\colon Col(v_i) \notin Avail(v_t)\ and\ i < t\}$
  - $Col(v_t) = any(Avail(v_t) - Hue(Bin(v_t)))$

- Avail(v_t) cannot equal to any bin hue, and by obsercation1, $Col(v_t)$ must be defined.
- Each round make progress by observation 2.
- By observation3, at least n/(k/2) bins uses. The number of colors is at most k, so performance ratio is at least $2n/k^2$, where $k\ = \log n$.