CSC2420: Algorithm Design, Analysis and Theory Spring 2019

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Week 1

Course Organization:

1. **Sources**: No one text; lots of sources including
   - various specialized graduate textbooks
   - my posted lecture notes (beware typos)
   - lecture notes from other Universities, and
   - research papers.

   See course web page.
2. **Lectures and Tutorials**: One two hour lecture per week with tutorials as needed and requested; not sure if we will have a TA.
3. **Office hours**: TBA but I always welcome questions (in class or otherwise). So feel free to drop by and/or email to schedule a time. My contact information: SF 2303B; bor@cs.toronto.edu. The course web page is www.cs.toronto.edu/~bor/2420s19

Course focus, disclaimer, and a confession

- The Design and Analysis of Algorithms is a very active field. Our course is a foundational course. However ....
- Disclaimer: We will not try to "cover" all aspects of the field but rather we will focus on some particular (but still reasonably diverse) topics relating to my research interests.
- In particular, my research interests are on combinatorial problems using mainly combinatorial algorithmic paradigms. And even more specifically, there is an emphasis on "*conceptually simple algorithms*". We will also consider a related theme, "*Beyond worst case analysis*".
- In my defense, perhaps most or all graduate algorithms courses are biased towards some research perspective.
- Confession: I am in the early stages of co-authoring (with Denis Pankratov) a new graduate text concerning online algorithms and "online-like" algorithms and we will make that accessible. Denis is teaching a course at Concordia that will parallel the text to a large extent and he will have lecture slides.

What is appropriate background? Grading

- Our undergraduate CSC 373 is essentially the prerequisite.
- Any of the popular undergraduate texts. For example, Kleinberg and Tardos; Cormen, Leiserson, Rivest and Stein; DasGupta, Papadimitriou and Vazirani.
- It certainly helps to have a good math background and in particular understand basic probability concepts (see Probability Primer), and some graph theory.

BUT any CS/ECE/Math graduate student (or mathematically oriented undergrad) should find the course accessible and useful.

- **Grading**: Will depend on how many students are taking this course for credit. In previous offerings there were three assignments with an occasional opportunity for some research questions. I am thinking that we may run part of this course as a reading course aligned with the new text depending on the number of students and interest in the material relating to the text with Denis.

Reviewing some basic algorithmic paradigms

We begin with some "conceptually simple" search/optimization algorithms.

The conceptually simplest "combinatorial" algorithms

Given an optimization problem, it seems to me that the conceptually simplest combinatorial approaches are:

Reviewing some basic algorithmic paradigms

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The conceptually simplest "combinatorial" algorithms

Given an optimization problem, it seems to me that the conceptually simplest combinatorial approaches are:

- brute force search
- divide and conquer
- online and greedy
- local search
- dynamic programming

Comment

- We usually dismiss brute force as it really isn't an interesting algorithmic approach but might work for small enough problems.
- Moreover, sometimes we can combine some aspect of brute force search with another approach as we will soon see.

Greedy algorithms in CSC373

Some of the greedy algorithms we study in different offerings of CSC 373

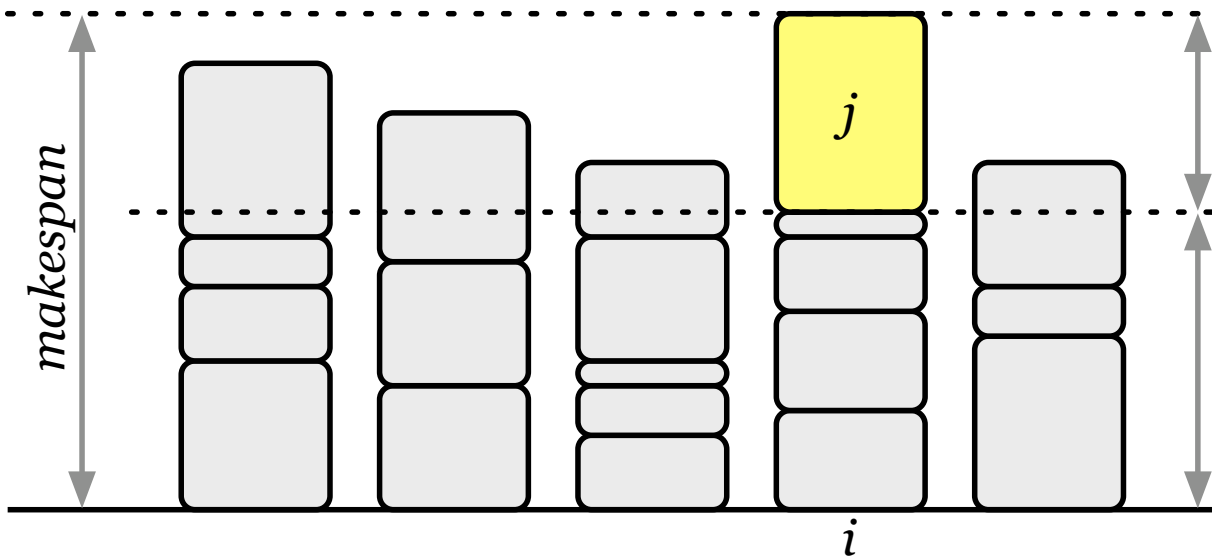
- The optimal algorithm for the fractional knapsack problem and the approximate algorithm for the proportional profit knapsack problem.
- The optimal unit profit interval scheduling algorithm and 3-approximation algorithm for proportional profit interval scheduling.
- The 2-approximate algorithm for the unweighted job interval scheduling problem and similar approximation for unweighted throughput maximization.
- Kruskal and Prim optimal algorithms for minimum spanning tree.
- Huffman's algorithm for optimal prefix codes.
- Graham's online and LPT approximation algorithms for makespan minimization on identical machines.
- The 2-approximation online algorithm for unweighted vertex cover via maximal matching.
- The "natural greedy" $\ln(m)$ approximation algorithm for set cover.

Greedy and online algorithms: Graham's *online* and LPT makespan algorithms

- Let's start with these two greedy algorithms that date back to 1966 and 1969 papers.
- These are good starting points since (preceding NP-completeness) Graham conjectured that makspan is a hard (requiring exponential time) problem to compute optimally but for which there were worst case approximation ratios (although he didn't use that terminology).
- This might then be called the start of worst case approximation algorithms. One could also even consider this to be the start of online algorithms and competitive analysis (although one usually refers to a 1985 paper by Sleator and Tarjan as the seminal paper in this regard).
- Moreover, there are some general concepts to be observed in this work and even after nearly 50 years, there are still open questions concerning the many variants of makespan problems.

The makespan problem for identical machines

- The input consists of n jobs $\mathcal{J} = J_1 \ldots, J_n$ that are to be scheduled on m identical machines.
- Each job J_k is described by a processing time (or load) p_k.
- The goal is to minimize the latest finishing time (maximum load) over all machines.
- That is, the goal is a mapping $\sigma : \{1, \ldots, n\} \to \{1, \ldots, m\}$ that minimizes $\max_k \left(\sum_{\ell:\sigma(\ell)=k} p_\ell \right)$.



[picture taken from Jeff Erickson's lecture notes]

Aside: The Many Variants of Online Algorithms

As I indicated, Graham's algorithm could be viewed as the first example of what has become known as *competitive analysis* (as named in a paper by Manasse, McGeoch and Sleator) following the paper by Sleator and Tarjan which explicitly advocated for this type of analysis. Another early (pre Sleator and Tarjan) example of such analysis was Yao's analysis of online bin packing algorithms.

In competitive analysis we compare the performance of an online algorithm against that of an optimal solution. The meaning of *online algorithm* here is that input items arrive sequentially and the algorithm must make an irrevocable decision concerning each item. (For makespan, an item is a job and the decision is to choose a machine on which the item is scheduled.)

But what determines the order of input item arrivals?

The Many Variants of Online Algorithms continued

- In the "standard" meaning of online algorithms (for CS theory), we think of an adversary as creating a nemesis input set and the ordering of the input items in that set. So this is traditional *worst case analysis* as in approximation algorithms applied to online algorithms. If not otherwise stated, we will assume this as the meaning of an online algorithm and if we need to be more precise we can say *online adversarial model*.
- We will also sometimes consider an *online stochastic model* where an adversary defines an input distribution and then input items are sequentially generated. There can be more general stochastic models (e.g., a Markov process) but the i.i.d model is common in analysis. Stochastic analysis as often seen in OR.
- In the i.i.d model, we can assume that the distribution is *known* by the algorithm or *unknown*.
- In the *random order model* (ROM), an adversary creates a size n nemesis input set and then the items from that set are given in a uniform random order (i.e. uniform over the $n!$ permutations)

Second aside: more general online frameworks

In the standard online model (and the variants we just mentioned), we are considering a one pass algorithm that makes one irrevocable decision for each input item.

There are many extensions of this one pass paradigm. For example:

- An algorithm is allowed some limited ability to revoke previous decisions.
- There may be some forms of lookahead (e.g. buffering of inputs).
- The algorithm may maintain a "small' number of solutions and then (say) take the best of the final solutions.
- The algorithm may do several passes over the input items.
- The algorithm may be given (in advance) some *advice bits* based on the entire input.

Throughout our discussion of algorithms, we can consider deterministic or randomized algorithms. In the online models, the randomization is in terms of the decisions being made. (Of course, the ROM model is an example of where the ordering of the inputs is randomized.)

A third aside: other measures of performance

The above variants address the issues of alternative input models, and relaxed versions of the online paradigm.

Competitive analysis is really just asymptotic approximation ratio analysis applied to online algorithms. Given the number of papers devoted to online competitive analysis, it is the standard measure of performance.

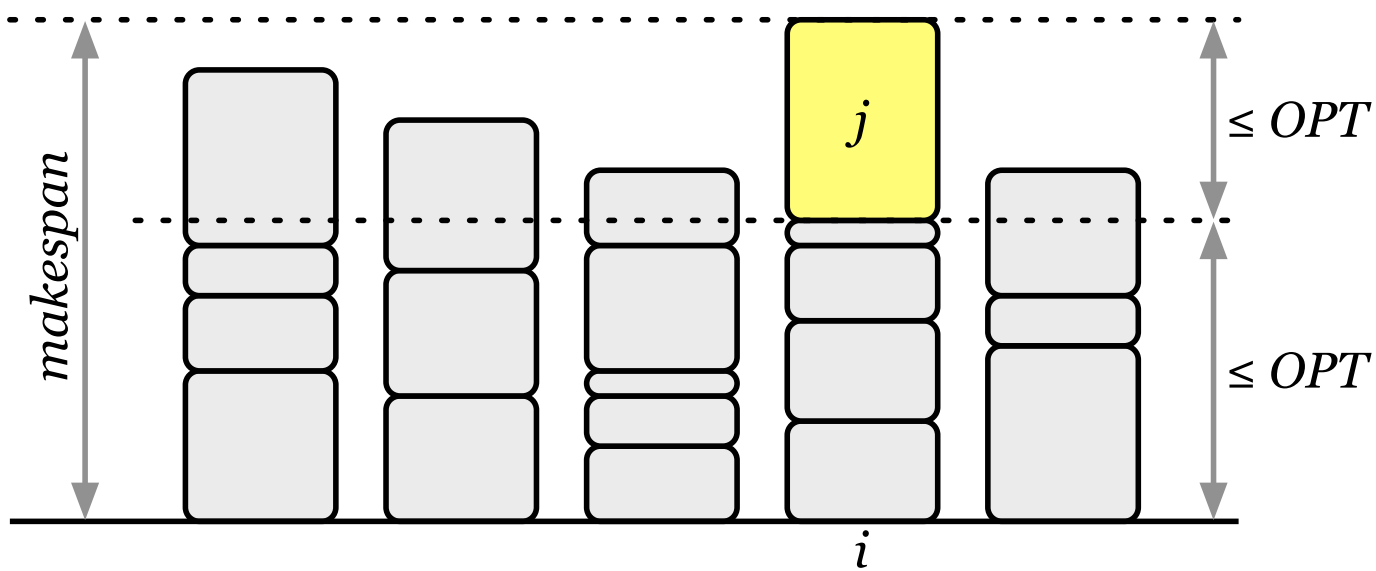
However, it has long been recognized that as a measure of performance, competitive analysis is often at odds with what seems to be observable in practice. Therefore, many alternative measures have been proposed. An overview of a more systematic study of alternative measures (as well as relaxed versions of the online paradigm and restricted input instances) for online algorithms is provided in Kim Larsen's lecture slides that I have placed on the course web site.

See, for example, the discussion of the *accommodating function* measure (for the dual bin packing problem), the *relative worst order* meaure for the bin packing coloring problem, and the page fault rate measure for paging.

Returning to Graham's online greedy algorithm

Consider input jobs in any order (e.g. as they arrive in an *online* setting) and schedule each job J_j on any machine having the least load thus far.

- We will see that the **approximation ratio** for this algorithm is $2 - \frac{1}{m}$; that is, for any set of jobs $\mathcal{J}$, $C_{Greedy}(\mathcal{J}) \leq (2 - \frac{1}{m}) C_{OPT}(\mathcal{J})$.
  - C_A denotes the cost (or makespan) of a schedule A.
  - OPT stands for any optimum schedule.
- **Basic proof idea:** $OPT \geq (\sum_j p_j)/m$; $OPT \geq max_j p_j$
  What is C_{Greedy} in terms of these requirements for any schedule?



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Graham's online greedy algorithm

Consider input jobs in any order (e.g. as they arrive in an online setting) and schedule each job J_j on any machine having the least load thus far.

- In the online "competitive analysis" literature the ratio $\frac{C_A}{C_{OPT}}$ is called the **competitive ratio** and it allows for this ratio to just hold in the limit as C_{OPT} increases. This is the analogy of *asymptotic approximation ratios*.

NOTE: Often, we will not provide proofs in the lecture notes but rather will do or sketch proofs in class (or leave a proof as an exercise).

- The approximation ratio for the online greedy is "tight" in that there is a sequence of jobs forcing this ratio.
- This bad input sequence suggests a better algorithm, namely the LPT (offline or sometimes called semi-online) algorithm.

Graham's LPT algorithm

Sort the jobs so that $p_1 \geq p_2 \ldots \geq p_n$ and then greedily schedule jobs on the least loaded machine.

- The (tight) approximation ratio of LPT is $\left(\frac{4}{3} - \frac{1}{3m}\right)$.
- It is believed that this is the **best** "greedy" algorithm but how would one prove such a result? This of course raises the question as to what is a greedy algorithm.
- We will present the priority model for greedy (and greedy-like) algorithms. I claim that all the algorithms mentioned on slide 6 can be formulated within the priority model.
- Assuming we maintain a priority queue for the least loaded machine,
  - the online greedy algorithm would have time complexity $O(n \log m)$ which is $(n \log n)$ since we can assume $n \geq m$.
  - the LPT algorithm would have time complexity $O(n \log n)$.

Partial Enumeration Greedy

- Combining the LPT idea with a brute force approach improves the approximation ratio but at a significant increase in time complexity.
- I call such an algorithm a "partial enumeration greedy" algorithm.

Optimally schedule the largest k jobs (for $0 \leq k \leq n$) and then greedily schedule the remaining jobs (in any order).

- The algorithm has approximation ratio no worse than $\left(1 + \frac{1-\frac{1}{m}}{1+\lfloor k/m \rfloor}\right)$.
- Graham also shows that this bound is tight for $k \equiv 0 \mod m$.
- The running time is $O(m^k + n \log n)$.
- Setting $k = \frac{1-\epsilon}{\epsilon} m$ gives a ratio of at most $(1 + \epsilon)$ so that *for any fixed* m, this is a PTAS (polynomial time approximation scheme). with time $O(m^{m/\epsilon} + n \log n)$.

Makespan: Some additional comments

- There are many refinements and variants of the makespan problem.
- There was significant interest in the best competitive ratio (in the online setting) that can be achieved for the identical machines makespan problem.
- The online greedy gives the best online ratio for m $= 2,3$ but better bounds are known for $m \geq 4$. For arbitrary m, as far as I know, following a series of previous results, the best known approximation ratio is 1.9201 (Fleischer and Wahl) and there is 1.88 inapproximation bound (Rudin). **Basic idea:** leave some room for a possible large job; this forces the online algorithm to be non-greedy in some sense but still within the online model.
- Randomization can provide somewhat better competitive ratios.
- Makespan has been actively studied with respect to three other machine models.

The uniformly related machine model

- Each machine i has a speed s_i
- As in the identical machines model, job J_j is described by a processing time or load p_j.
- The processing time to schedule job J_j on machine i is p_j/s_i.
- There is an online algorithm that achieves a constant competitive ratio.
- I think the best known deterministic (resp. randomized) online ratio is 5.828 i(resp. 4.311) due to P. Berman et al [2000] following the first constant ratio by Aspnes et al.
- Ebenlendr and Sgall [2015] establish a deterministic online inapproximation of 2.564 following the 2.438 deterministic online inapproximation of Berman et al. who also proved a 1.8372 inapproximation for any randomized online algorithm.

The restricted machines model

- Every job J_j is described by a pair (p_j, S_j) where $S_j \subseteq \{1, \ldots, m\}$ is the set of machines on which J_j can be scheduled.
- This (and the next model) have been the focus of a number of papers (for both online and offline) and there has been some relatively recent progress in the offline restricted machines case.
- Even for the case of two allowable machines per job (i.e. the *graph orientation problem*), this is an interesting problem and we will look at some recent work later.
- Azar et al show that $\log_2(m)$ (resp. $\ln(m)$) is (up to ± 1) the best competitive ratio for deterministic (resp. randomized) online algorithms with the upper bounds obtained by the "natural greedy algorithm".
- It is not known if there is an offline greedy-like algorithm for this problem that achieves a constant approximation ratio. Regev [IPL 2002] shows an $\Omega(\frac{\log m}{\log \log m})$ inapproximation for "fixed order priority algorithms" for the restricted case when every job has 2 allowable machines.

The unrelated machines model

- This is the most general of the makespan machine models.
- Now a job J_j is represented by a vector $(p_{j,1}, \ldots, p_{j,m})$ where $p_{j,i}$ is the time to process job J_j on machine i.
- A classic result of Lenstra, Shmoys and Tardos [1990] shows how to solve the (offline) makespan problem in the unrelated machine model with approximation ratio 2 using LP rounding.
- There is an online algorithm with approximation $O(\log m)$. Currently, this is the best approximation known for greedy-like (e.g. priority) algorithms even for the restricted machines model although there has been some progress made in this regard (which we will discuss later).
- NOTE: All statements about what we will do later should be understood as intentions and not promises.

Makespan with precedence constraints; how much should we trust our intuition

Graham also considered the makespan problem on identical machines for jobs satisfying a precedence constraint. Suppose $\prec$ is a partial ordering on jobs meaning that if $J_i \prec J_k$ then J_i must complete before J_k can be started. Assuming jobs are ordered so as to respect the partial order (i.e., can be reordered within the priority model) Graham showed that the ratio $2 - \frac{1}{m}$ is achieved by "the natural greedy algorithm", call it $\mathcal{G}_\prec$.

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Graham's 1969 paper is entitled "Bounds on Multiprocessing Timing Anomalies" pointing out some very non-intuitive anomalies that can occur.

Consider $\mathcal{G}_{\prec}$ and suppose we have a given an input instance of the *makespan with precedence* problem. Which of the following should never lead to an increase in the makepan objective for the instance?

- Relaxing the precedence $\prec$
- Decreasing the processing time of some jobs
- Adding more machines

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- Decreasing the processing time of some jobs
- Adding more machines

In fact, all of these changes could increase the makespan value.

The knapsack problem

The $\{0,1\}$ knapsack problem

- Input: Knapsack size capacity C and n items $\mathcal{I} = \{I_1, \ldots, I_n\}$ where $I_j = (v_j, s_j)$ with v_j (resp. s_j) the profit value (resp. size) of item I_j.
- Output: A feasible subset $S \subseteq \{1, \ldots, n\}$ satisfying $\sum_{j \in S} s_j \leq C$ so as to maximize $V(S) = \sum_{j \in S} v_j$.

Note: I would prefer to use approximation ratios $r \geq 1$ (so that we can talk unambiguously about upper and lower bounds on the ratio) but many people use approximation ratios $\rho \leq 1$ for maximization problems; i.e. $ALG \geq \rho OPT$. For certain topics, this is the convention.

- It is easy to see that the most natural greedy methods (sort by non-increasing profit densities $\frac{v_j}{s_j}$, sort by non-increasing profits v_j, sort by non-decreasing size s_j) will not yield any constant ratio.

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- It is easy to see that the most natural greedy methods (sort by non-increasing profit densities $\frac{v_j}{s_j}$, sort by non-increasing profits v_j, sort by non-decreasing size s_j) will not yield any constant ratio.
- Can you think of nemesis sequences for these three greedy methods?
- What other orderings could you imagine?

The partial enumeration greedy PTAS for knapsack

The $PGreedy_k$ Algorithm

Sort $\mathcal{I}$ so that $\frac{v_1}{s_1} \geq \frac{v_2}{s_2} \ldots \geq \frac{v_n}{s_n}$
For every feasible subset $H \subseteq \mathcal{I}$ with $|H| \leq k$
    Let $R = \mathcal{I} - H$ and let OPT_H be the optimal solution for H
    Consider items in R (in the order of profit densities)
    and greedily add items to OPT_H not exceeding knapsack capacity C.
        % It is sufficient for bounding the approximation ratio to stop
            as soon as an item is too large to fit
End For
Output: the OPT_H having maximum profit.

Sahni's PTAS result

Theorem (Sahni 1975): $V(OPT) \leq (1 + \frac{1}{k})V(PGreedy_k)$.

- This algorithm takes time kn^k and setting $k = \frac{1}{\epsilon}$ yields a $(1 + \epsilon)$ approximation running in time $\frac{1}{\epsilon}n^{\frac{1}{\epsilon}}$.
- An *FPTAS* is an algorithm achieving a $(1 + \epsilon)$ approximation with running time $poly(n, \frac{1}{\epsilon})$. There is an FPTAS for the knapsack problem (using dynamic programming and scaling the input values) so that the PTAS algorithm for knapsack was quickly subsumed. But still the partial enumeration technique is a general approach that is often useful in trying to obtain a PTAS (e.g. as mentioned for makespan).
- This technique (for $k = 3$) was also used by Sviridenko to achieve an $\frac{e}{e-1} \approx 1.58$ approximation for monotone submodular maximization subject to a knapsack constraint. It is NP-hard to do better than a $\frac{e}{e-1}$ approximation for submodular maximization subject to a cardinality constraint (i.e. when all knapsack sizes are 1).
- Usually such inapproximations are more precisely stated as "NP-hard to achieve $\frac{e}{e-1} + \epsilon$ for any $\epsilon > 0$".

The priority algorithm model and variants

As part of our discussion of greedy (and greedy-like) algorithms, I want to present the priority algorithm model and how it can be extended in (conceptually) simple ways to go beyond the power of the priority model.

- What is the intuitive nature of a greedy algorithm as exemplified by the CSC 373 algorithms we mentioned? With the exception of Huffman coding (which we can also deal with), like online algorithms, all these algorithms consider one input item in each iteration and make an irrevocable "greedy" decision about that item..
- We are then already assuming that the class of search/optimization problems we are dealing with can be viewed as making a decision D_k about each input item I_k (e.g. on what machine to schedule job I_k in the makespan case) such that $\{(I_1, D_1), \ldots, (I_n, D_n)\}$ constitutes a feasible solution.
- For online problems (where the adversary determines the ordering of input item), the abstract problem formulation is called *request-answer games*. Note: The *line-search problem* and other online navigational search problems are not request-answer games.

Priority model continued

- Note: that a problem is only fully specified when we say how input items are represented. (This is usually implicit in an online algorithm.)
- We mentioned that a "non-greedy" online algorithm for identical machine makespan can improve the competitive ratio; that is, the algorithm does not always place a job on the (or a) least loaded machine (i.e. does not make a greedy or locally optimal decision in each iteration). It isn't always obvious if or how to define a "greedy" decision but for many problems the definition of greedy can be informally phrased as "live for today" (i.e. assume the current input item could be the last item) so that the decision should be an optimal decision given the current state of the computation.

Greedy decisions and priority algorithms continued

- For example, in the knapsack problem, a greedy decision always takes an input if it fits within the knapsack constraint and in the makespan problem, a greedy decision always schedules a job on some machine so as to minimize the increase in the makespan. (This is somewhat more general than saying it must place the item on the least loaded machine.)
- If we do not insist on greediness, then priority algorithms would best have been called myopic algorithms.
- We have both fixed order priority algorithms (e.g. unweighted interval scheduling and LPT makespan) and adaptive order priority algorithms (e.g. the set cover greedy algorithm and Prim's MST algorithm).
- The key concept is to indicate how the algorithm chooses the order in which input items are considered. We cannot allow the algorithm to choose say "an optimal ordering".
- We might be tempted to say that the ordering has to be determined in polynomial time but that gets us into the "tarpit" of trying to prove what can and can't be done in (say) polynomial time.

The priority model definition

- We take an information theoretic viewpoint in defining the orderings we allow.
- Lets first consider deterministic fixed order priority algorithms. Since I am using this framework mainly to argue negative results (e.g. a priority algorithm for the given problem cannot achieve a stated approximation ratio), we will view the semantics of the model as a game between the algorithm and an adversary.
- Initially there is some (possibly infinite) set $\mathcal{J}$ of potential inputs. The algorithm chooses a total ordering π on $\mathcal{J}$. Then the adversary selects a subset $\mathcal{I} \subset \mathcal{J}$ of actual inputs so that $\mathcal{I}$ becomes the input to the priority algorithm. The input items $I_1, \ldots, I_n$ are ordered according to π.
- In iteration k for $1 \leq k \leq n$, the algorithm considers input item I_k and based on this input and all previous inputs and decisions (i.e. based on the current state of the computation) the algorithm makes an irrevocable decision D_k about this input item.

The fixed (order) priority algorithm template

$\mathcal{J}$ is the set of all possible input items
Decide on a total ordering π of $\mathcal{J}$
Let $\mathcal{I} \subset \mathcal{J}$ be the input instance
$S := \varnothing$ % S is the set of items already seen
$i := 0$ % $i = |S|$
while $\mathcal{I} \setminus S \neq \varnothing$ **do**
    $i := i + 1$
    $\mathcal{I} := \mathcal{I} \setminus S$
    $I_i := \min_\pi \{I \in \mathcal{I}\}$
    make an irrevocable decision D_i concerning I_i
    $S := S \cup \{I_i\}$
end

Figure: The template for a fixed priority algorithm

Some comments on the priority model

- A special (but usual) case is that π is determined by a function $f : \mathcal{J} \to \Re$ and and then ordering the set of actual input items by increasing (or decreasing) values $f()$. (We can break ties by say using the input identifier of the item to provide a total ordering of the input set.) N.B. We make no assumption on the complexity or even the computability of the ordering π or function f.
- NOTE: Online algorithms are fixed order priority algorithms where the ordering is given *adversarially*; that is, the items are ordered by the input identifier of the item.
- As stated we do not give the algorithm any additional information other than what it can learn as it gradually sees the input sequence.
- However, we can allow priority algorithms to be given some (hopefully easily computed) global information such as the number of input items, or say in the case of the makespan problem the minimum and/or maximium processing time (load) of any input item. (Some inapproximation results can be easily modified to allow such global information.)

The adaptive priority model template

$\mathcal{J}$ is the set of all possible input items
$\mathcal{I}$ is the input instance
$S := \varnothing$ % S is the set of items already considered
$i := 0$ % $i = |S|$
while $\mathcal{I} \setminus S \neq \varnothing$ **do**
    $i := i + 1$
    decide on a total ordering π_i of $\mathcal{J}$
    $\mathcal{I} := \mathcal{I} \setminus S$
    $I_i := \min_{\leq_{\pi_i}} \{I \in \mathcal{I}\}$
    make an irrevocable decision D_i concerning I_i
    $S := S \cup \{I_i\}$
    $\mathcal{J} := \mathcal{J} \setminus \{I : I \leq_{\pi_i} I_i\}$
    % some items cannot be in input set
end

Figure: The template for an adaptive priority algorithm

Inapproximations with respect to the priority model

Once we have a precise model, we can then argue that certain approximation bounds are not possible within this model. Such inapproximation results have been established with respect to priority algorithms for a number of problems but for some problems much better approximations can be established using extensions of the model.

1. For the weighted interval selection (a *packing problem*) with arbitrary weighted values (resp. for proportional weights $v_j = |f_j - s_j|$), no priority algorithm can achieve a constant approximation (respectively, better than a 3-approximation).
2. For the knapsack problem, no priority algorithm can achieve a constant approximation. We note that the maximum of two greedy algorithms (sort by value, sort by value/size) is a 2-approximation.
3. For the set cover problem, the natural greedy algorithm is essentially the best priority algorithm.
4. As previously mentioned, for deterministic fixed order priority algorithms, there is an $\Omega(\log m / \log \log m)$ inapproximation bound for the makespan problem in the restricted machines model.

More on provable limitations of the priority model

The above mentioned inapproximations are with respect to deterministic priority algorithms. For an adaptive algorithm, the game between an algorithm and an adversary can conceptually be naturally viewed an alternating sequence of actions;

- The adversary eliminates some possible input items
- The algorithm makes a decision for the item with highest priority and chooses a new ordering for all possible remaining input items.

However, we note that for deterministic algorithms, since the adversary knows precisely what the algorithm will do in each iteation, it could initially set the input $\mathcal{I}$ once the algorithm is known.

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For randomized algorithms, there is a difference between an *oblivious adversary* that creates an initial subset $\mathcal{I}$ of items vs an *adaptive adversary* that is playing the game adaptively reacting to each decision by the algorithm. Why?

Unless stated otherwise we usually analyze randomized algorithms (for any type of algorithm) with respect to an oblivious adversary.

Extensions of the priority order model

In discussing more general online frameworks, we already implicitly suggested some extensions of the basic priority model (that is, the basic model where we have one-pass and one irrevocable decision). The following online or priority algorithm extensions can be made precise:

- Decisions can be *revocable* to some limited extent or at some cost. For example, we know that in the basic priority model we cannot achieve a constant approximation for weighted interval scheduling. However, if we are allowed to permanently discard previously accepted intervals (while always maintaining a feasible solution), then we can achieve a 4-approximation. (but provably not optimality).
- While the knapsack problem cannot be approximated to within any constant, we can achieve a 2-approximation by taking the maximum of 2 greedy algorithms. More generally we can consider some "small" number k of priority (or online) algorithms and take the best result amongst these k algorithms. The partial enumeration greedy algorithm for the makespan and knapsack problems are an example of this type of extension.

Extensions of the priority order model continued

- Closely related to the "best of k online" model is the concept of online algoitthms with "advice". (One could also study priority algorithms with advice but that has not been done to my knowledge.) There are two advice models, a model where one measures the maximum number of advice bits per input item, and a model where we are given some number ℓ of advice bits at the start of the computation. The latter model is what I will mean by "online with advice." Online with ℓ advice bits is equivalent to the max of $k = 2^\ell$ online model.

Extensions of the priority order model continued

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  NOTE: This model is a very permissive in that the advice bits can be a function of the entire input. Of course, in practice we want these advice bits to be "easily determined" (e.g., the number of input items, or the ratio of the largest to smallest weight/value) but in keeping with the information theoretic perspective of onine and priority algorithms, one doesn't impose any such restriction.
- There are more general parallel priority based models than "best of k" algorithms. Namely, parallel algorithms could be spawning or aborting threads (as in the pBT model to be discussed later).

Multipass algorithms

- Another model that provides improved results is to allow multiple passes (over the input items) rather than just one pass.
- This is not a well studied model but there are two relatively new noteworthy results that we will be discussing:
  1. There is deterministic 3/4 approximation for weighted Max-Sat that is achieved by two "online passes" (i.e., the input sequence is determined by an adversary) over the input sequence whereas there is evidence that no one pass deterministic online or priority algorithm can acheive this ratio.
  2. There is a $\frac{3}{5}$ approximation for biparitie matching that is achieved by two online passes whereas no deterministic online or priority algorithm can do asymptotically better than a $\frac{1}{2}$ approximation.
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- It is not clear how best to formalize these multi-pass algorithms. Why? What information should we be allowed to convey between passes?

Greedy algorithms for the set packing problem

One of the new areas in theoretical computer science is algorithmic game theory and mechanism design and, in particular, auctions including what are known as *combinatorial auctions*. The underlying combinatorial problem in such auctions is the set packing problem.

The set packing problem

We are given n subsets $S_1, \ldots, S_n$ from a universe U of size m. In the weighted case, each subset S_i has a weight w_i. The goal is to choose a disjoint subcollection $\mathcal{S}$ of the subsets so as to maximize $\sum_{S_i \in \mathcal{S}} w_i$. In the s-set packing problem we have $|S_i| \leq s$ for all i.

- This is a well studied problem and by reduction from the max clique problem, there is an $m^{\frac{1}{2}-\epsilon}$ hardness of approximation assuming $NP \neq ZPP$. For s-set packing with constant $s \geq 3$, there is an $\Omega(s/\log s)$ hardness of approximation assuming $P \neq NP$.
- We will consider two "natural" greedy algorithms for the s-set packing problem and a non obvious greedy algorithm for the set packing problem. These greedy algorithms are all fixed order priority.

The first natural greedy algorithm for set packing

Greedy-by-weight (*Greedy*$_{wt}$

Sort the sets so that $w_1 \geq w_2 \ldots \geq w_n$.
$\mathcal{S} := \emptyset$
For $i : 1 \ldots n$
    If S_I does not intersect any set in $\mathcal{S}$ then
        $\mathcal{S} := \mathcal{S} \cup S_i$.
End For

- In the unweighted case (i.e. $\forall i, w_i = 1$), this is an online algorithm.
- In the weighted (and hence also unweighted) case, greedy-by-weight provides an s-approximation for the s-set packing problem.
- The approximation bound can be shown by a charging argument where the weight of every set in an optimal solution is charged to the first set in the greedy solution with which it intersects.

The second natural greedy algorithm for set packing

Greedy-by-weight-per-size

Sort the sets so that $w_1/|S_1| \geq w_2/|S_2| \ldots \geq w_n/|S_n|$.
$\mathcal{S} := \varnothing$
For $i : 1 \ldots n$
    If S_I does not intersect any set in $\mathcal{S}$ then
        $\mathcal{S} := \mathcal{S} \cup S_i$.
End For

- In the weighted case, greedy-by-weight provides an s-approximation for the s-set packing problem.
- For both greedy algorithms, the approximation ratio is tight; that is, there are examples where this is essentially the approximation. In particular, these algorithms only provide an m-approximation where $m = |U|$.
- We usually assume $n >> m$ and note that by just selecting the set of largest weight, we obtain an n-approximation. So the goal is to do better than $\min\{m, n\}$.

Improving the approximation for set packing

- In the unweighted case, greedy-by-weight-per-size can be restated as sorting so that $|S_1| \leq |S_2| \ldots \leq |S_n|$ and it can be shown to provide an $\sqrt{m}$-approximation for set packing.
- On the other hand, greedy-by-weight-per-size does not improve the m-approximation for weighted set packing.

Greedy-by-weight-per-squareroot-size

Sort the sets so that $w_1/\sqrt{|S_1|} \geq w_2/\sqrt{|S_2|} \ldots \geq w_n/\sqrt{|S_n|}$.
$\mathcal{S} := \varnothing$
For $i : 1 \ldots n$
    If S_I does not intersect any set in $\mathcal{S}$ then
        $\mathcal{S} := \mathcal{S} \cup S_i$.
End For

Theorem: Greedy-by-weight-per-squareroot-size provides a $2\sqrt{m}$-approximation for the set packing problem. And as noted earlier, this is asymptotically the best possible approximation assuming $NP \neq ZPP$.

Another way to obtain an $O(\sqrt{m})$ approximation

There is another way to obtain the same aysmptototic improvement for the weighted set packing problem. Namely, we can use the idea of partial enumeration greedy; that is somehow combining some kind of brute force (or naive) approach with a greedy algorithm.

Partial Enumeration with Greedy-by-weight ($PGreedy_k$)

Let Max_k be the best solution possible when restricting solutions to those containing at most k sets. Let G be the solution obtained by $Greedy_{wt}$ applied to sets of cardinality at most $\sqrt{m/k}$. Set $PGreedy_k$ to be the best of Max_k and G.

- Theorem: $PGreedy_k$ achieves a $2\sqrt{m/k}$-approximation for the weighted set packing problem (on a universe of size m)
- In particular, for $k = 1$, we obtain a $2\sqrt{m}$ approximation and this can be improved by an arbitrary constant factor $\sqrt{k}$ at the cost of the brute force search for the best solution of cardinality k; that is, at the cost of say n^k.