It is certainly preferable for you to solve the questions without consulting a published source. However, if you are using a published source then you must specify the source and you should try to improve upon the presentation of the result.

If you would like to discuss any questions with someone else that is fine BUT at the end of any collaboration you must spend at least one hour playing video games or watching two periods of Maple Leaf hockey or maybe even start reading a good novel before writing anything down.

Unless stated otherwise, all subquestions (1(i), 1(ii), 1(iii), 1(iv), etc) are worth 10 points. If you do not know how to answer a question, state “I do not know how to answer this (sub) question” and you will receive 20% (i.e. 2 of 10 points) for doing so. You can receive partial credit for any reasonable attempt to answer a question BUT no credit for arguments that make no sense.

In class I can clarify any questions you may have about this assignment.
1. Consider the makespan problem for the identical machines model with \( m \) machines. We sketched the proof that the worst case competitive ratio \( \frac{C_{\text{Greedy}}}{C_{\text{OPT}}} \) of the natural online greedy algorithm is \( \leq 2 - \frac{1}{m} \). We also gave an example showing that this ratio is “tight” (i.e. cannot be improved) for this algorithm.

(i) Finish the proof that was sketched in Lecture 1; that is, use the fact that \( C_{\text{OPT}} \geq \sum_{1 \leq i \leq n} p_i / m \) for any sequence of \( n \) input jobs and \( C_{\text{OPT}} \geq p_i \) where job \( J_i \) has “load” \( p_i \).

Note: You can easily find the proof of this result but you should try to solve it without searching for a solution.

(ii) Argue for \( m = 2 \) (resp. \( m = 3 \)) machines that any (not necessarily greedy) online algorithm would have competitive ratio no better than \( \frac{3}{2} \) (resp. \( \frac{5}{3} \)) so that the natural greedy online algorithm approximation is tight for \( m = 2 \) and \( m = 3 \) for any online algorithm.

(iii) Consider greedy online algorithm for the makespan problem but now in the random order ROM model. Show that for any \( \epsilon > 0 \), there exists a sufficiently large \( m \) such that the (expected) approximation ratio \( \frac{E[C_{\text{Greedy}}]}{C_{\text{OPT}}} \geq 2 - \epsilon \). Here the expectation is with respect to the uniform distribution on input arrival order.

If you cannot prove the stated claim then prove any ratio greater than 1. Hint: generalize the nemesis sequence for the adversarial competitive ratio.

(iv) Consider the LPT algorithm for the makespan problem. The LPT approximation ratio is \( \frac{4}{3} - \frac{1}{3m} \). The outline of the proof is as follows:

i. Without loss of generality, the job causing the makespan is \( p_r \), the job having minimum processing cost.

ii. Recall the two facts about bounds for OPT that was used to prove the approximation for the online Greedy algorithm. That is, the makespan is always at least \( \max_i p_i \) and at least \( \sum_{i=1}^m T_i / m \) where \( T_i \) is the makespan (i.e., completion time) of machine \( i \).

iii. Use these two facts to show that if the stated approximation bound does not hold, then \( p_r > OPT/3 \).
iv. It follows that OPT can only schedule at most 2 jobs per machine.

v. Show how to transform an OPT schedule into the LPT schedule without increasing the makespan and thereby deriving a contradiction.

Fill in the details for the above proof outline.

2. Consider the following algorithm for the set packing problem. Let $S = \{S_1, S_2, \ldots, S_n\}$ be the input instance with $S_i \subseteq \{1, 2, \ldots, m\}$.

- Partition $S = S^1 \cap S^2$ where $S^1 = \{S_i : |S_i| \leq \sqrt{m}\}$.
- Let $R^1$ be the result of running the greedy algorithm $Greedy_{wt}$ on $S^1$ and let $R^2$ be the set having the maximum weight of sets in $S^2$.
- Return the better of $R^1$ and $R^2$.

Show that $2\sqrt{m}$ is a bound on the approximation ratio of this set packing algorithm.

3. More to follow