

Fine-Grained Complexity and Algorithm Design Boot Camp

Parameterized Reductions

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Lower bounds

So far we have seen positive results: basic algorithmic techniques for fixed-parameter tractability.

What kind of negative results we have?

- Can we show that a problem (e.g., **CLIQUE**) is **not** FPT?
⇒ Today
- Can we show that a problem (e.g., **VERTEX COVER**) has **no** algorithm with running time, say, $2^{o(k)} \cdot n^{O(1)}$?
⇒ Thursday 3pm

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This would require showing that $P \neq NP$: if $P = NP$, then, e.g., k -**CLIQUE** is polynomial-time solvable, hence FPT.

Can we give some evidence for negative results?

Goals of this talk

Two goals:

- 1 Explain the theory behind parameterized intractability.
- 2 Show examples of parameterized reductions.

Parameterized complexity

To build a complexity theory for parameterized problems, we need two concepts:

- An appropriate notion of reduction.
- An appropriate hypothesis.

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Polynomial-time reductions are not good for our purposes.

Example: Graph G has an independent set k if and only if it has a vertex cover of size $n - k$.

⇒ Transforming an **INDEPENDENT SET** instance (G, k) into a **VERTEX COVER** instance $(G, n - k)$ is a correct polynomial-time reduction.

However, **VERTEX COVER** is FPT, but **INDEPENDENT SET** is not known to be FPT.

Parameterized reduction

Definition

Parameterized reduction from problem P to problem Q : a function ϕ with the following properties:

- $\phi(x)$ is a yes-instance of $Q \iff x$ is a yes-instance of P .
- $\phi(x)$ can be computed in time $f(k) \cdot |x|^{O(1)}$, where k is the parameter of x ,
- If k is the parameter of x and k' is the parameter of $\phi(x)$, then $k' \leq g(k)$ for some function g .

Fact: If there is a parameterized reduction from problem P to problem Q and Q is FPT, then P is also FPT.

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Fact: If there is a parameterized reduction from problem P to problem Q and Q is FPT, then P is also FPT.

Non-example: Transforming an **INDEPENDENT SET** instance (G, k) into a **VERTEX COVER** instance $(G, n - k)$ is **not** a parameterized reduction.

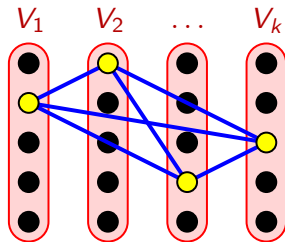
Example: Transforming an **INDEPENDENT SET** instance (G, k) into a **CLIQUE** instance (\overline{G}, k) is a parameterized reduction.

MULTICOLORED CLIQUE

A useful variant of **CLIQUE**:

MULTICOLORED CLIQUE: The vertices of the input graph G are colored with k colors and we have to find a clique containing one vertex from each color.

(or **PARTITIONED CLIQUE**)



Theorem

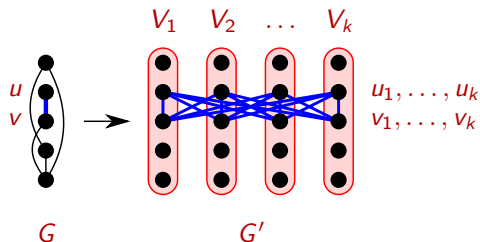
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MULTICOLORED CLIQUE

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Create G' by replacing each vertex v with k vertices, one in each color class. If u and v are adjacent in the original graph, connect all copies of u with all copies of v .



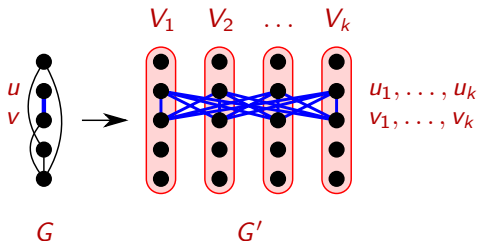
k -clique in $G \iff$ multicolored k -clique in G' .

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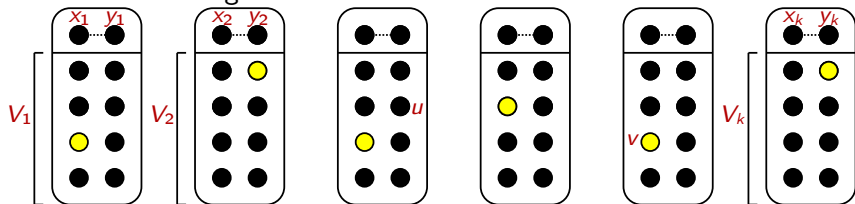
Similarly: reduction to **MULTICOLORED INDEPENDENT SET**.

DOMINATING SET

Theorem

There is a parameterized reduction from **MULTICOLORED INDEPENDENT SET** to **DOMINATING SET**.

Proof: Let G be a graph with color classes V_1, \dots, V_k . We construct a graph H such that G has a multicolored k -clique iff H has a dominating set of size k .



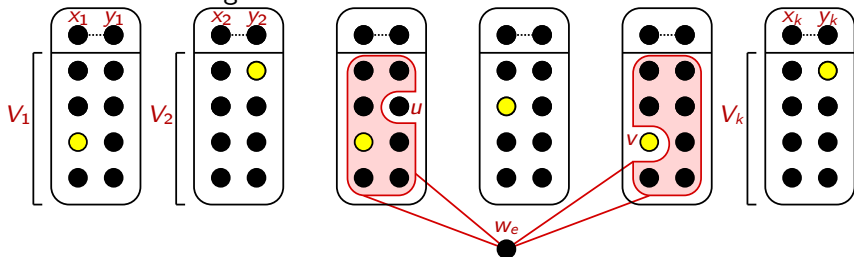
- The dominating set has to contain one vertex from each of the k cliques V_1, \dots, V_k to dominate every x_i and y_i .

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- The dominating set has to contain one vertex from each of the k cliques V_1, \dots, V_k to dominate every x_i and y_i .
- For every edge $e = uv$, an additional vertex w_e ensures that these selections describe an independent set.

Variants of DOMINATING SET

- **DOMINATING SET**: Given a graph, find k vertices that dominate every vertex.
- **RED-BLUE DOMINATING SET**: Given a bipartite graph, find k vertices on the red side that dominate the blue side.
- **SET COVER**: Given a set system, find k sets whose union covers the universe.
- **HITTING SET**: Given a set system, find k elements that intersect every set in the system.

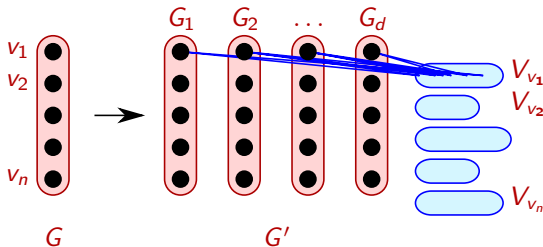
All of these problems are equivalent under parameterized reductions, hence at least as hard as **CLIQUE**.

Regular graphs

Theorem

There is a parameterized reduction from **CLIQUE** to **CLIQUE** on regular graphs.

Proof: Given a graph G and an integer k , let d be the maximum degree of G . Take d copies of G and for every $v \in V(G)$, fully connect every copy of v with a set V_v of $d - d(v)$ vertices.



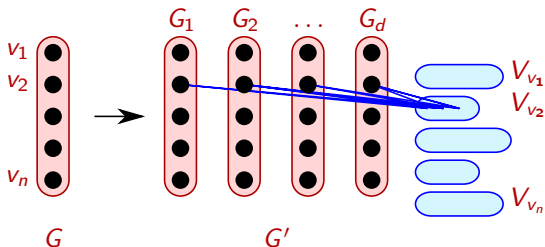
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PARTIAL VERTEX COVER

PARTIAL VERTEX COVER: Given a graph G , integers k and s , find k vertices that cover at least s edges.

Theorem

There is a parameterized reduction from **INDEPENDENT SET** on regular graphs parameterized by k to **PARTIAL VERTEX COVER** parameterized by k .

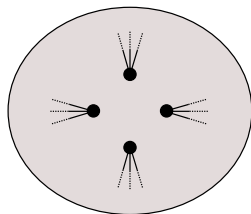
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Proof: If G is d -regular, then k vertices can cover $s := kd$ edges if and only if there is an independent set of size k .



$$d = 3, k = 4, s = 12$$

Hard problems

Hundreds of parameterized problems are known to be at least as hard as **CLIQUE**:

- **INDEPENDENT SET**
- **SET COVER**
- **HITTING SET**
- **CONNECTED DOMINATING SET**
- **INDEPENDENT DOMINATING SET**
- **PARTIAL VERTEX COVER** parameterized by k
- **DOMINATING SET** in bipartite graphs
- ...

We believe that none of these problems are FPT.

Basic hypotheses

It seems that parameterized complexity theory cannot be built on assuming $P \neq NP$ – we have to assume something stronger.

Let us choose a basic hypothesis:

Engineers' Hypothesis

k -CLIQUE cannot be solved in time $f(k) \cdot n^{O(1)}$.

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Exponential Time Hypothesis (ETH)

n -variable 3SAT cannot be solved in time $2^{o(n)}$.

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INDEPENDENT SET \Rightarrow Turing machines

Theorem

There is a parameterized reduction from INDEPENDENT SET to the k -STEP HALTING PROBLEM.

Proof: Given a graph G and an integer k , we construct a Turing machine M and an integer $k' = O(k^2)$ such that M halts in k' steps if and only if G has an independent set of size k .

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The alphabet Σ of M is the set of vertices of G .

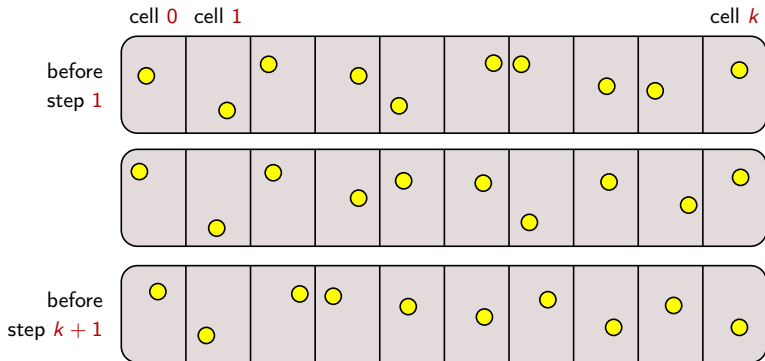
- In the first k steps, M nondeterministically writes k vertices to the first k cells.
- For every $1 \leq i \leq k$, M moves to the i -th cell, stores the vertex in the internal state, and goes through the tape to check that every other vertex is nonadjacent with the i -th vertex (otherwise M loops).
- M does k checks and each check can be done in $2k$ steps $\Rightarrow k' = O(k^2)$.

Turing machines \Rightarrow INDEPENDENT SET

Theorem

There is a parameterized reduction from the k -STEP HALTING PROBLEM to INDEPENDENT SET.

Proof: Given a Turing machine M and an integer k , we construct a graph G that has an independent set of size $k' := (k + 1)^2$ if and only if M halts in k steps.



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Proof: Given a Turing machine M and an integer k , we construct a graph G that has an independent set of size $k' := (k + 1)^2$ if and only if M halts in k steps.

- G consists of $(k + 1)^2$ cliques, thus a k' -independent set has to contain one vertex from each.
- The selected vertex from clique $K_{i,j}$ describes the situation before step i at cell j : what is written there, is the head there, and if so, what the state is, and what the next transition is.
- We add edges between the cliques to rule out inconsistencies: head is at more than one location at the same time, wrong character is written, head moves in the wrong direction etc.

Summary

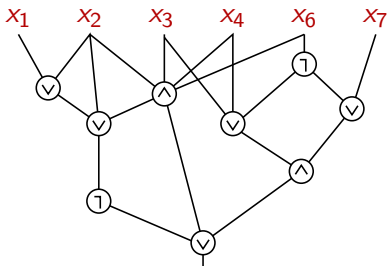
- INDEPENDENT SET and k -STEP HALTING PROBLEM can be reduced to each other \Rightarrow Engineers' Hypothesis and Theorists' Hypothesis are equivalent!
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Summary

- INDEPENDENT SET and k -STEP HALTING PROBLEM can be reduced to each other \Rightarrow Engineers' Hypothesis and Theorists' Hypothesis are equivalent!
- INDEPENDENT SET and k -STEP HALTING PROBLEM can be reduced to DOMINATING SET.
- Is there a parameterized reduction from DOMINATING SET to INDEPENDENT SET?
- Probably not. Unlike in NP-completeness, where most problems are equivalent, here we have a hierarchy of hard problems.
 - INDEPENDENT SET is $W[1]$ -complete.
 - DOMINATING SET is $W[2]$ -complete.
- Does not matter if we only care about whether a problem is FPT or not!

Boolean circuit

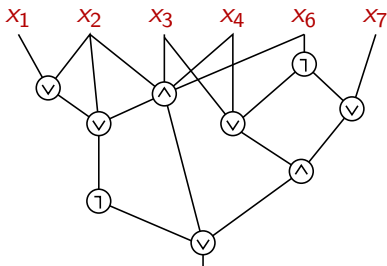
A **Boolean circuit** consists of input gates, negation gates, AND gates, OR gates, and a single output gate.



CIRCUIT SATISFIABILITY: Given a Boolean circuit C , decide if there is an assignment on the inputs of C making the output true.

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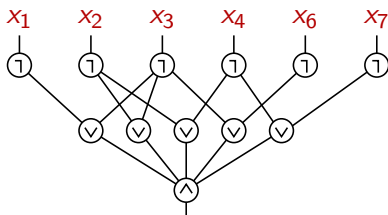
CIRCUIT SATISFIABILITY: Given a Boolean circuit C , decide if there is an assignment on the inputs of C making the output true.

Weight of an assignment: number of true values.

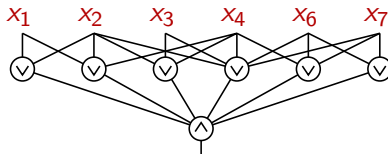
WEIGHTED CIRCUIT SATISFIABILITY: Given a Boolean circuit C and an integer k , decide if there is an assignment of weight k making the output true.

WEIGHTED CIRCUIT SATISFIABILITY

INDEPENDENT SET can be reduced to WEIGHTED CIRCUIT SATISFIABILITY:

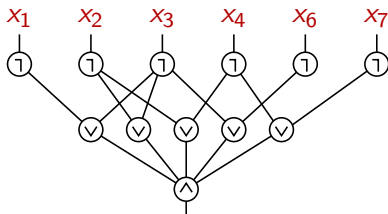


DOMINATING SET can be reduced to WEIGHTED CIRCUIT SATISFIABILITY:

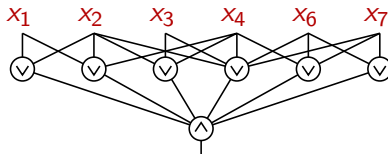


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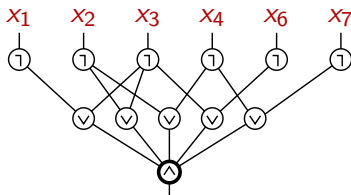
To express DOMINATING SET, we need more complicated circuits.

Depth and weft

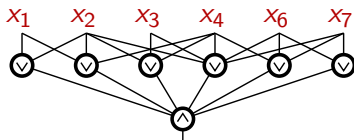
The **depth** of a circuit is the maximum length of a path from an input to the output.

A gate is **large** if it has more than 2 inputs. The **weft** of a circuit is the maximum number of large gates on a path from an input to the output.

INDEPENDENT SET: weft 1, depth 3



DOMINATING SET: weft 2, depth 2



The W-hierarchy

Let $C[t, d]$ be the set of all circuits having weft at most t and depth at most d .

Definition

A problem P is in the class $W[t]$ if there is a constant d and a parameterized reduction from P to **WEIGHTED CIRCUIT SATISFIABILITY** of $C[t, d]$.

We have seen that **INDEPENDENT SET** is in $W[1]$ and **DOMINATING SET** is in $W[2]$.

Fact: **INDEPENDENT SET** is $W[1]$ -complete.

Fact: **DOMINATING SET** is $W[2]$ -complete.

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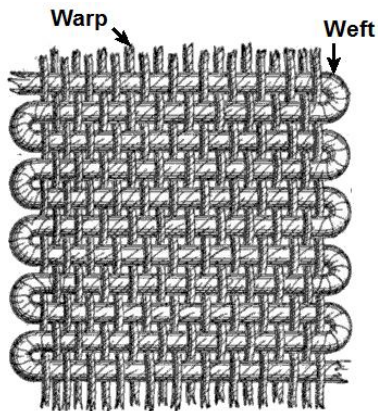
Fact: **DOMINATING SET** is $W[2]$ -complete.

If any $W[1]$ -complete problem is FPT, then $FPT = W[1]$ and every problem in $W[1]$ is FPT.

If any $W[2]$ -complete problem is in $W[1]$, then $W[1] = W[2]$.

\Rightarrow If there is a parameterized reduction from **DOMINATING SET** to **INDEPENDENT SET**, then $W[1] = W[2]$.

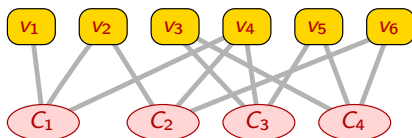
Weft



Weft is a term related to weaving cloth: it is the thread that runs from side to side in the fabric.

Parameterized reductions

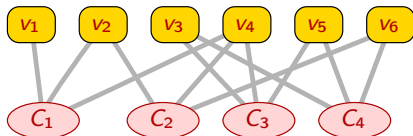
Typical NP-hardness proofs: reduction from e.g., CLIQUE or 3SAT, representing each vertex/edge/variable/clause with a gadget.



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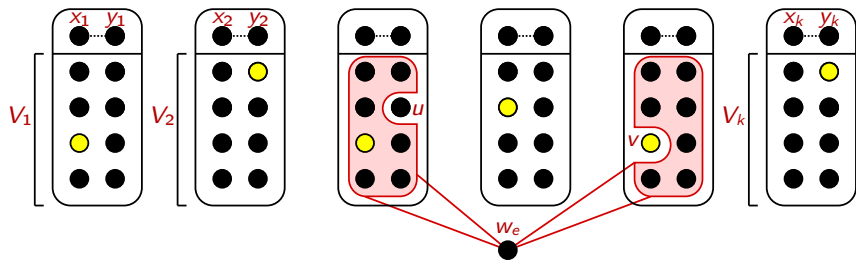
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Types of parameterized reductions:

- Reductions keeping the structure of the graph.
 - CLIQUE \Rightarrow INDEPENDENT SET
 - INDEPENDENT SET on regular graphs \Rightarrow PARTIAL VERTEX COVER
- Reductions with vertex representations.
 - MULTICOLORED INDEPENDENT SET \Rightarrow DOMINATING SET
- Reductions with vertex and edge representations.

Vertex representation

Recall: Reduction from **MULTICOLORED INDEPENDENT SET** to **DOMINATING SET**:



LIST COLORING

LIST COLORING is a generalization of ordinary vertex coloring:
given a

- graph G ,
- a set of colors C , and
- a list $L(v) \subseteq C$ for each vertex v ,

the task is to find a coloring c where $c(v) \in L(v)$ for every v .

Theorem

VERTEX COLORING is FPT parameterized by treewidth.

However, list coloring is more difficult:

Theorem

LIST COLORING is $W[1]$ -hard parameterized by treewidth.

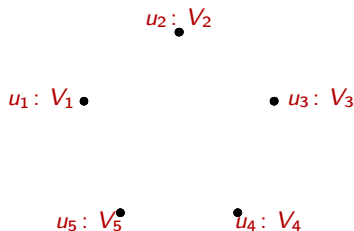
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Proof: By reduction from MULTICOLORED INDEPENDENT SET.

- Let G be a graph with color classes V_1, \dots, V_k .
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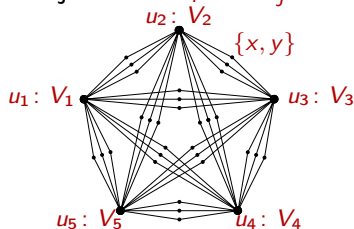
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- If $x \in V_i$ and $y \in V_j$ are adjacent in G , then we need to ensure that $c(u_i) = x$ and $c(u_j) = y$ are not true at the same time \Rightarrow we add a vertex adjacent to u_i and u_j whose list is $\{x, y\}$.



Vertex representation

Key idea

- Represent the k vertices of the solution with k gadgets.
- Connect the gadgets in a way that ensures that the represented values are **compatible**.

But sometimes it is very difficult to create connections that force two gadgets to be compatible. . .

ODD SET

ODD SET: Given a set system \mathcal{F} over a universe U and an integer k , find a set S of at most k elements such that $|S \cap F|$ is odd for every $F \in \mathcal{F}$.

Theorem

ODD SET is $W[1]$ -hard parameterized by k .

ODD SET

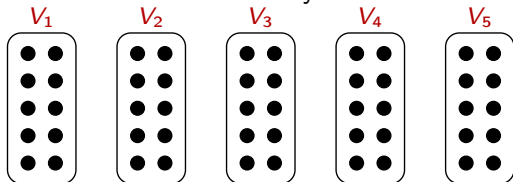
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First try: Reduction from MULTICOLORED INDEPENDENT SET.

Let $U = V_1 \cup \dots \cup V_k$ and introduce each set V_i into \mathcal{F} .

\Rightarrow The solution has to contain exactly one element from each V_i .



If $xy \in E(G)$, how can we express that $x \in V_i$ and $y \in V_j$ cannot be selected simultaneously?

ODD SET

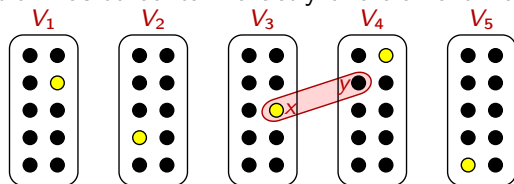
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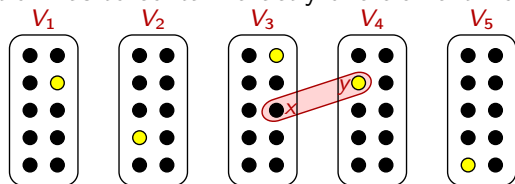
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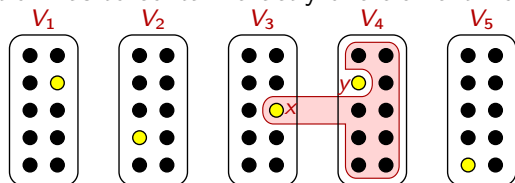
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ODD SET is $W[1]$ -hard parameterized by k .

First try: Reduction from MULTICOLORED INDEPENDENT SET.

Let $U = V_1 \cup \dots \cup V_k$ and introduce each set V_i into \mathcal{F} .

\Rightarrow The solution has to contain exactly one element from each V_i .



If $xy \in E(G)$, how can we express that $x \in V_i$ and $y \in V_j$ cannot be selected simultaneously? Seems difficult:

- introducing $\{x, y\}$ into \mathcal{F} forces that **exactly one** of x and y appears in the solution,
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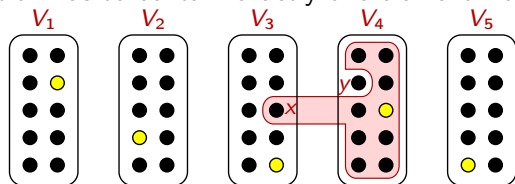
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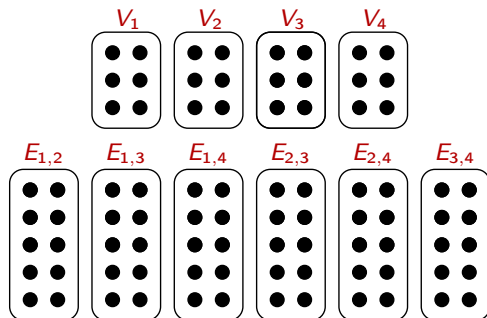
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ODD SET

Reduction from **MULTICOLORED CLIQUE**.

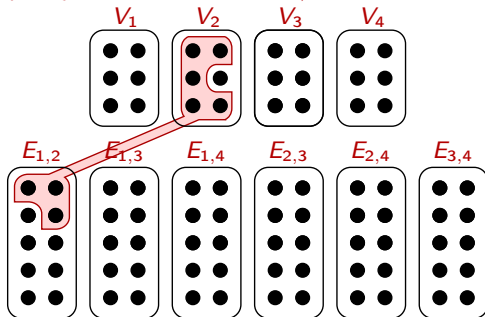
- $U := \bigcup_{i=1}^k V_i \cup \bigcup_{1 \leq i < j \leq k} E_{i,j}$.
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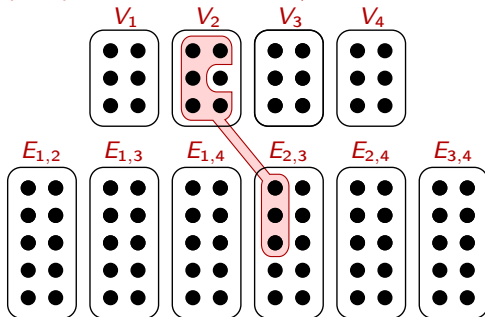
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($V_i \setminus \{v\}$) \cup {every edge from $E_{i,x}$ with endpoint v }
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ODD SET

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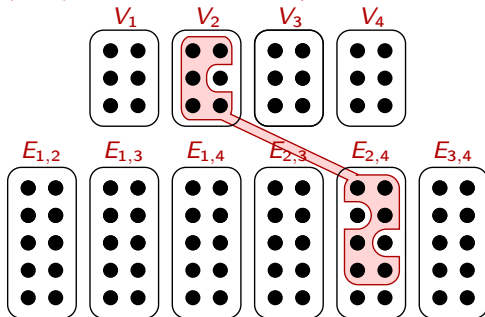
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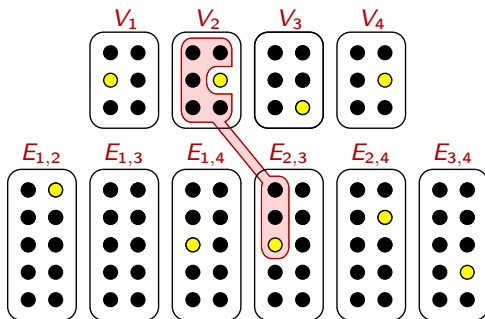
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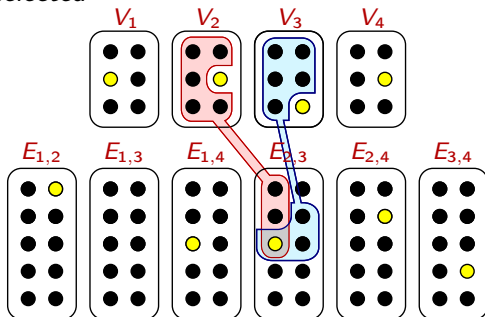
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- $v \in V_i$ selected \iff edges with endpoint v are selected from $E_{i,x}$ and $E_{x,i}$
- $v_i \in V_i$ selected \iff edge $v_i v_j$ is selected in $E_{i,x}$
 $v_j \in V_j$ selected



Vertex and edge representation

Key idea

- Represent the vertices of the clique by k gadgets.
- Represent the edges of the clique by $\binom{k}{2}$ gadgets.
- Connect edge gadget $E_{i,j}$ to vertex gadgets V_i and V_j such that if $E_{i,j}$ represents the edge between $x \in V_i$ and $y \in V_j$, then it **forces** V_i to x and V_j to y .

The connection between the edge gadget and a vertex gadget needs to express a simple projection relation: a selection of an edge forces a selection of a vertex.

Typically blows up the parameter to $O(k^2)$!

Variants of ODD SET

The following problems are $W[1]$ -hard:

- ODD SET
- EXACT ODD SET (find a set of size exactly k ...)
- EXACT EVEN SET
- UNIQUE HITTING SET
(at most k elements that hit each set exactly once)
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Open question:

? **EVEN SET:** Given a set system \mathcal{F} and an integer k , find a nonempty set S of at most k elements such $|F \cap S|$ is even for every $F \in \mathcal{F}$.

Summary

- By parameterized reductions, we can show that lots of parameterized problems are at least as hard as **CLIQUE**, hence unlikely to be fixed-parameter tractable.
- Connection with Turing machines gives some supporting evidence for hardness (only of theoretical interest).
- The **W**-hierarchy classifies the problems according to hardness (only of theoretical interest).
- Important trick in **W[1]**-hardness proofs: vertex and edge representations.