On the Power of Graph Searching

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Overview

- Overview of Graph Searching
- Maximal (Maximum?) Independent Set Algorithm
- Cocomparability Graphs, Comparability Graphs and Posets
- Comments, New Results and Concluding Remarks

Overview of Graph Searching

- Algorithms for visiting all vertices of a given graph.
- BFS and DFS discovered in the late 1890s for maze traversal.
- Desire for simple, efficient, easily implementable algorithms.
- In the 1960s and 1970s BFS and DFS were shown to have many applications in computer science.
- In 1976 Rose, Tarjan and Lueker presented LBFS as a way of recognizing chordal graphs (no induced cycle of size greater than 3).
- Many applications of LBFS were later found including easier linear time algorithms for modular and split decomposition.
- There is a vertex ordering characterization of LBFS orderings [Golumbic; Brandstadt, Dragan and Nicolai].
- The study of such VOCs lead to the discovery of LDFS [C. and Krueger].

Maximal (Maximum?) Independent Set Algorithm

Maximal (Maximum?) Independent Set (MIS):
Input: A connected graph $G = (V, E)$ and vertex ordering σ
Output: Set I containing the vertices of an IS

$I \leftarrow \emptyset$; $V' \leftarrow V$ $\{V'$ stores the unprocessed vertices$\}$; $j \leftarrow 0$;

while $V' \neq \emptyset$ **do**
    $j \leftarrow j + 1$;
    $x_j \leftarrow$ the rightmost vertex of V', as ordered by σ;
    $I \leftarrow I \cup \{x_j\}$; $V' \leftarrow V' \setminus N[x_j]$;
end

return (I)

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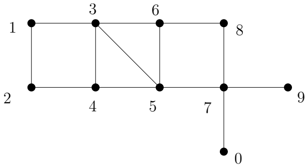
- How could we "Certify" that I is a maximum IS?
- *Find a clique cover of the same cardinality. Note that for any graph G, $\alpha(G) \leq \kappa(G)$ where α is the maximum cardinality IS and κ is the minimum cardinality clique cover. If the graph is perfect, then equality holds.*

Example



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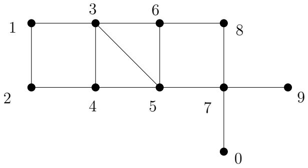

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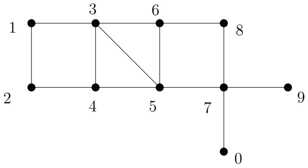

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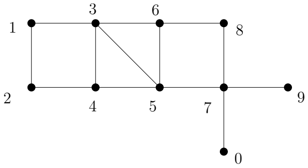

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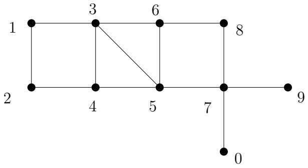

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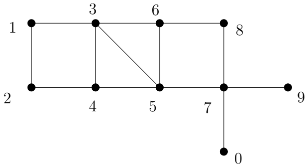

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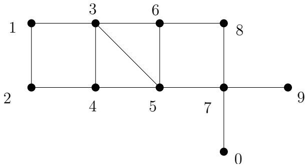

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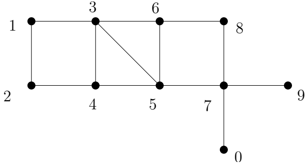

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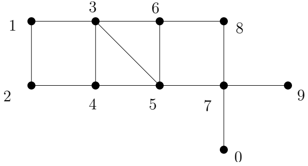

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- Note that I is of maximum cardinality, as shown by the clique cover (from right to left): $\{2, 1\}, \{5, 3, 4\}, \{8, 6\}, \{7, 9\}, \{0\}$

Cocomparability Graphs, Comparability Graphs and Posets

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- A comparability graph together with an acyclic transitive orientation of its edges can be equivalently represented by a linear extension of partially ordered set (also called a **poset**).
- A poset consists of a set V together with an irreflexive, antisymmetric and transitive binary relation $<$ that imposes a "precedes" relationship on certain pairs of elements of V. Two elements $x, y \in V$ are said to be **comparable** if $x < y$, or $y < x$; otherwise the elements are called **incomparable**. A **linear extension** of a poset is a total ordering of V that respects the ordering of all comparable pairs.

Definitions Continued

- DEFINITION: $G(V, E)$ is a interval graph if it is the intersection graph of subpaths of a path; namely, each vertex represents a subpath and two vertices are adjacent iff their subpaths intersect. Note that interval graphs are a strict subset of cocomparability graphs.

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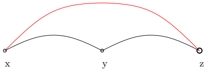
- THEOREM: G is cocomparability iff there is an ordering of V such that for all $x < y < z, xz \in E$ implies $xy \in E$ OR $yz \in E$ OR both (COCOMP order)



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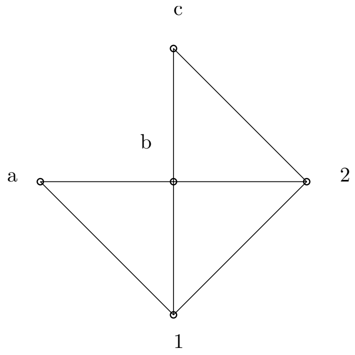
- Note that from the poset perspective, this algorithm is computing a maximum sized set of mutually comparable elements.

Lexicographic Depth First Search (LDFS)

Roughly speaking, LDFS is a DFS where ties are broken by favouring vertices with adjacencies to most recently visited vertices.

COMPLETE the LDFS



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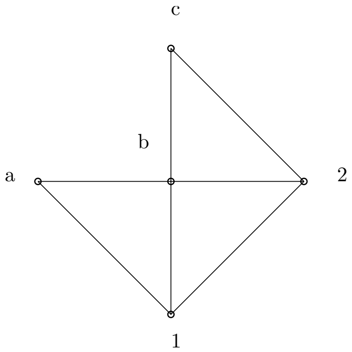

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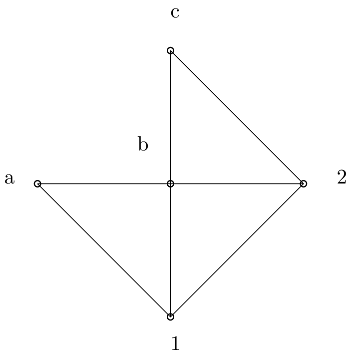

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IMPLEMENTATION: $O(min\{n^2, n + mloglogn\})$ Spinrad and ???

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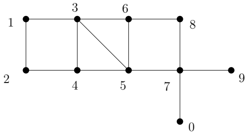
- A $^{+}$ sweep breaks ties by choosing the rightmost tied vertex as ordered by τ.

Example of LDFS$^+$



Consider this graph and cocomp order $\tau = 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 0$

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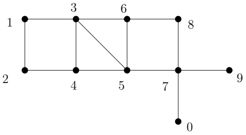

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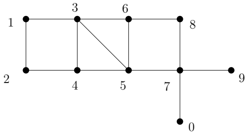

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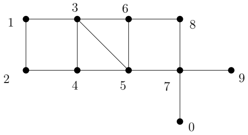

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- The next vertex is 6 - now LDFS forces 5 and then 3, followed by $4, 2, 1$.

Comments

- By having a certification step, we will either guarantee that we have a maximum IS or will output a message that the given ordering is not a cocomp ordering. Note that we do not confirm that our given ordering τ is a cocomp ordering.
- THEOREM (C. + K.):
  An ordering σ is an LDFS ordering iff for all $a <_\sigma b <_\sigma c$ where $ac \in E$, $ab \notin E$, there exists $a <_\sigma d <_\sigma b$ such that $db \in E, dc \notin E$.
- Similar LDFS$^+$ modified interval graph algorithms work for:
  - Minimum Path Cover (equivalent to the bump number problem on posets) [C., Dalton, H.]
  - Longest Path [Mertzios, C.]
- These algorithms give us insight into the "LDFS structure of posets".

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- Using these results we have a new graph search and simple algorithms to compute minimal clique separators and to find simplicial vertices in cocomp graphs.
- Very recently Lalla Mouatadid and E.K. have found a linear time algorithm to find the maximum **weighted** independent set in a cocomp graph. They have also shown how to find an LDFS cocomp order in linear time, given an arbitrary cocomp order.

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- Can these results extend to AT-free graphs?
- Can we use graph searching for heuristic algorithms? Already used for the diameter of the giant component of the Facebook Graph.

Thank you for your attention