Lecture 9

Sublinear Time Algorithms

Sublinear Time Algorithms

- Sublinear time
  - Algorithm runs in $o(n)$ time, where n = length of input
    - Assume direct access to i^{th} bit of the input
  - Algorithm cannot even read the entire input!
  - (With a few exceptions) the algorithm must
    - Use randomization
    - Provide an approximately accurate answer

- Also interesting: sublinear space
  - Algorithm uses $o(n)$ *additional* space

Motivation

- Huge datasets
  - World-wide web, social networks, genome project, sales logs, census data, high-resolution images, fine-grained scientific measurements, …
- Need fast algorithms for subroutines that will be called frequently
- Many sublinear algorithms turn out to be streaming algorithms, which only need to access incoming data once

Exact vs Inexact Algorithms

- Exact: Always provides the right answer
- Inexact: Provides an approximately optimal answer
  - ANS = right answer, ALG = output of algorithm
  - For numerical answers (e.g., counting problems)
    - $(1-\epsilon)\, ANS \leq ALG \leq (1+\epsilon)\, ANS$
  - For binary answers (e.g., yes/no problems)
    - 1-sided error:
      - $ANS = YES \Rightarrow ALG = YES$ with probability 1
      - $ANS = NO \Rightarrow ALG = NO$ with probability $\geq\ ^2/_3$
    - 2-sided error:
      - $ALG = ANS$ with probability $\geq\ ^2/_3$

Exact vs Inexact Algorithms

- Exact: Always provides the right answer
- Inexact: Provides an approximately optimal answer
  - ANS = right answer, ALG = output of algorithm
  - For "property testing"
    - Property satisfied ⇒ $ALG = YES$ with probability 1
    - Property at least ϵ-far from being satisfied ⇒ $ALG = NO$ with probability ≥ ${}^{2}/_{3}$
    - Property not satisfied, but ϵ-close to being satisfied ⇒ don't care what ALG is
    - Will see this later…

Deterministic + Exact

- Always give the right answer using a deterministic algorithm that does not read the entire input!
  - Seems impossible

- You already know one such algorithm
  - Binary search! $O(\log n)$ time, deterministic, exact.
  - Needs to assume that input is already sorted.

- In general, deterministic + exact is impossible unless input is structured.

Deterministic + Inexact

- Approximating the diameter in a metric space
  - Points $x_1, \ldots, x_n$, distance metric d
  - Input: n^2 numbers $d_{ij} = d(x_i, x_j)$
  - Goal: estimate diameter $D = \max_{i,j} d(x_i, x_j)$
- Algorithm: Pick arbitrary x_i, return $D_i = \max_{j \neq i} d_{ij}$
- Analysis:
  - $O(n)$ running time, "sublinear" in the input length $O(n^2)$
  - $D = d_{a,b} \leq d_{a,i} + d_{i,b} \leq D_i + D_i = 2D_i$
  - Also, clearly $D \geq D_i$. Thus, we have a 2-approximation!

Deterministic + Inexact

- This is also somewhat an exception
- Note: If you were given n explicit points in a Euclidean space instead of the $O(n^2)$ distances, then $O(n)$ would no longer be sublinear.
- For most sublinear time algorithms, randomization is a must!

Randomized + Exact

- Known as Las Vegas algorithms
  - Distribution over deterministic algorithms
  - Each algorithm is exact, i.e., produces the correct answer.
  - The algorithms have varying costs on different instances.
  - Hope is that a randomization over them will have low expected cost on every instance.
- Example: two algos {Alg1,Alg2}, two instances {I1,I2}
  - Alg1 takes 1000 steps on I1, 10 on I2.
  - Alg2 takes 10 steps on I2, 1000 on I2.
  - (½)Alg1+(½)Alg2 takes 505 expected steps on any instance

Searching in Sorted List

- Input: A sorted doubly linked list with n elements.
  - Imagine you have an array A with $O(1)$ access to $A[i]$
  - $A[i]$ is a tuple (x_i, p_i, n_i)
    - Value,
    - index of previous element in list,
    - index of next element in list.
  - Sorted: $x_{p_i} \leq x_i \leq x_{n_i}$
- Task: Given x, check if there exists i s.t. $x = x_i$
- Goal: We will give a randomized + exact algorithm with expected running time $O(\sqrt{n})$!

Searching in Sorted List

- Motivation:
  - Often we deal with large datasets that are stored in a large file on disk, or possibly broken into multiple files
  - Creating a new, sorted version of the dataset is expensive
  - It is often preferred to "implicitly sort" the data by simply adding previous-next pointers along with each element
  - Would like algorithms that can operate on such implicitly sorted versions and yet achieve sublinear running time
    - Just like binary search achieves for an explicitly sorted array

Searching in Sorted List

Algorithm:

- Select $\sqrt{n}$ random indices R
- Access x_j for each $j \in R$
- Find the nearest $x_j : j \in R$ on each side of x
  - $p \in R$ such that $x_p = \max\{x_j : x_j \leq x, j \in R\}$
  - $q \in R$ such that $x_q = \min\{x_j : x_j > x, j \in R\}$
  - One of the two must exist (WHY?).
- If p exists, start at $A[p]$, and keep going next until you discover x, or you reach $A[q]$ or end of list.
- If q exists, start at $A[q]$, and keep going back until you discover x, or you reach $A[p]$ or beginning of list.

Searching in Sorted List

- Analysis:
  - Take arbitrary value x. Take the minimum value x_i in the list that is at least x. The algorithm is searching for x_i.
  - The algorithm throws $\sqrt{n}$ random "darts" on the list.
  - Chernoff bounds: the probability that there is no dart in $c\sqrt{n}$ elements to the left (resp. right) of x_i is $2^{-\Omega(c)}$.
  - So, the expected distance of x_i to the dart on its left (and its right) is $O(\sqrt{n})$.
  - The algorithm finds these two darts in $O(\sqrt{n})$ time, and uses $O(\sqrt{n})$ search to locate x_i.

Searching in Sorted List

- Theorem: There exists an exact randomized algorithm for searching in a sorted doubly linked list in $O(\sqrt{n})$ expected running time.
  - Note: We don't *really* require the list to be doubly linked. Just "next" pointer suffices if we have a pointer to the first element of the list (a.k.a. "anchored list").

- We can also prove optimality of this algorithm!
- Theorem: No exact randomized algorithm can do this in $o(\sqrt{n})$ expected running time.

Yao's Principle

- Proves a *lower bound* on the expected running time of the best exact randomized algorithm
  - "The expected time of a randomized algorithm R on the worst input I is no better than the expected time taken, under the worst probability distribution $\mathbb{I}$ over inputs, by the best deterministic algorithm A for that distribution."
  - $\max_I E_R[C(R,I)] \geq \max_D \min_A E_{I\sim D}[C(A,I)]$
- Special case of von Neumann's minimax theorem for two-player zero-sum games
  - Can see a randomized algorithm as a distribution over all deterministic algorithms

Searching in Sorted List

- Adversarial distribution: uniform random ordering of n distinct values
- Goal: Search the max value (last element of the list)
- We want to show that any deterministic algorithm takes $\Omega(\sqrt{n})$ steps in expectation.
- Deterministic algorithms have only two operations:
  - Op A: Access next/previous of an already accessed element.
  - Op B: Compute an index k, and access the k^{th} element.

Searching in Sorted List

- Note: In a B op, the algorithm can compute index k using any information it has until that point.

- Let T = the set of last $\sqrt{n}$ elements on the list.
  - If the algorithm doesn't access any element of T using a B op, it must take $\Omega(\sqrt{n})$ A ops to locate the last element starting from any accessed element $\Rightarrow$ done!

- We show that the algorithm requires $\Omega(\sqrt{n})$ steps in expectation to access an element of T using B op
  - Note: Each B op returns a random element from the yet unexplored list. (WHY?)

Searching in Sorted List

- To show: $\Omega(\sqrt{n})$ steps required to access T
  - After a many A ops and b many B ops, probability of accessing an element of T in the next B op is at most
$$\frac{|T|}{|\text{Unexplored List}|} = \frac{\sqrt{n}}{n-a-b} \le \frac{\sqrt{n}+a+b}{n}$$
  - Probability that no element of T is accessed after a many A ops and b many B ops is at least $\left(1-\frac{\sqrt{n}+a+b}{n}\right)^b$
  - This implies that the expected number of steps until an element of T is accessed is $\Omega(\sqrt{n})$. (WHY?) QED!

Sublinear Geometric Algorithms

- Chazelle, Liu, and Magen [2003] proved the $\Theta(\sqrt{n})$ bound for searching in a sorted linked list
  - Their main focus was to generalize these ideas to come up with sublinear algorithms for geometric problems
  - Polygon intersection: Given two convex polyhedra, check if they intersect.
  - Point location: Given a Delaunay triangulation (or Voronoi diagram) and a point, find the cell in which the point lies.
  - They provided optimal $O(\sqrt{n})$ algorithms for both these problems.

Randomized + Inexact

- We will now move to inexact algorithms that return approximately accurate answers.

- Let us start with a counting problem where the answer is numerical.

Estimating Avg Degree in Graph

- Input: Graph G with n vertices, and access to an oracle that returns the degree of a queried vertex in $O(1)$ time.

- Ouptut: α-approximation of the average degree d of the vertices of G.
  - α-approximation $\Rightarrow$ answer lies in $[{}^{d}/_{\alpha}, \alpha \cdot d]$

- Goal: $(2+\epsilon)$-approximation in expected time $O\left(\epsilon^{-O(1)}\sqrt{n}\right)$
  - ϵ is constant $\Rightarrow$ sublinear in input size n

Estimating Avg Degree in Graph

- Wait!
  - Isn't this equivalent to "given an array of n numbers between 1 and $n - 1$, estimate their average"?
  - No! That requires $\Omega(n)$ time for constant approximation!
    - Consider an instance with constantly many $n - 1$'s, and all other 1's: you may not discover any $n - 1$ until you query $\Omega(n)$ numbers
  - Why are degree sequences more special?
    - Erdős–Gallai theorem: $d_1 \geq \cdots \geq d_n$ is a degree sequence iff their sum is even and $\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} d_i$.
    - Intuitively, we will sample $O(\sqrt{n})$ vertices
      - We may not discover the few high degree vertices, but we'll find their neighbors, and thus account for their edges anyway!

Estimating Avg Degree in Graph

- **Algorithm:**
  - Take ${}^{8}/_{\epsilon}$ random subsets $S_i \subseteq V$ with $|S_i| = s$
  - Compute the average degree d_{S_i} in each S_i.
  - Return $\widehat{d} = \min_i d_{S_i}$
- **Analysis:**
  - We will show that with $s = \Theta\left(\sqrt{n}/\epsilon^{O(1)}\right)$, we can ensure $\widehat{d} \in \left[\left({}^{1}/_{2} - \epsilon\right) d, (1+\epsilon)\, d\right]$ with probability at least ¾.
    - Note: #queries (and running time) = $O\left({}^{\sqrt{n}}/_{\epsilon^{O(1)}}\right)$
    - Feige [2006] improved this to $O\left(\epsilon^{-1}\sqrt{{}^{n}/_{d_0}}\right)$ if we know $d \geq d_0$
      - In particular, even with $d_0 = 1$, we have $O\left({}^{\sqrt{n}}/_{\epsilon}\right)$ queries.

Estimating Avg Degree in Graph

- Claim 1: We can choose $s = \Theta\left(\sqrt{n}/\epsilon^{O(1)}\right)$ such that $\Pr[d_S < (1/2 - \epsilon)\, d] \le \epsilon/64$.
- Proof:
  - Let H be the set of $\sqrt{\epsilon' n}$ highest degree vertices in G, and $L = V \backslash \mathrm{H}$.
  - Sub-claim: $\sum_{i \in L} d_i \ge (1/2 - \epsilon') \sum_{i \in V} d_i$
    - Note that $\sum_{i \in V} d_i$ counts each edge in the graph twice.
    - $\sum_{i \in L} d_i$ might omit at most $\epsilon' n$ edges within H, and might only count edges between H and L once.
      - Thus, $\sum_{i \in L} d_i \ge 1/2 \left(\sum_{i \in V} d_i - \epsilon' n\right)$
    - The sub-claim now follows when you substitute $n \le \sum_{i \in V} d_i$ in the above equation (which is true because G is connected).

Estimating Avg Degree in Graph

- Proof:
  - We proved: $\sum_{i \in L} d_i \geq (1/2 - \epsilon') \sum_{i \in V} d_i$
    - Thus, average degree in $L \geq (1/2 - \epsilon')\, d$.
  - A *lower bound* on d_S: assume all its vertices come from L
    - Let d_H = *minimum* degree of any vertex in H.
    - Let X_i = degree of i^{th} vertex in $S \Rightarrow X_i \in [1, d_H]$
    - $E[X_i] \geq (1/2 - \epsilon')\, d \geq (1/2 - \epsilon') d_H\, |H|/n$
    - $t = E[\sum_{i=1}^{s} X_i] = \Omega(d_H)$ due to our choice of s
  - Hoeffding's bound:
    - $\Pr[\sum_{i=1}^{s} X\_i < (1 - \epsilon') \cdot t] \leq e^{-\frac{t\,(\epsilon')^2}{d_H}} \leq \frac{\epsilon}{64}$
    - Set ϵ' such that $(1 - \epsilon') \cdot (1/2 - \epsilon') = 1/2 - \epsilon$

Estimating Avg Degree in Graph

- Claim 2: $\Pr[d_S > (1+\epsilon)d] \leq 1 - {}^{\epsilon}/_{2}$.
- Proof:
  - Markov's inequality
  - $\Pr[d_S > \ell] \leq \frac{E[d_S]}{\ell} = \frac{d}{(1+\epsilon)\, d} = \frac{1}{1+\epsilon} \leq 1 - \frac{\epsilon}{2}$
- Finishing the proof:
  - $\Pr[d_S < ({}^{1}/_{2} - \epsilon)\, d] \leq {}^{\epsilon}/_{64}$ -- low probability!
  - $\Pr[d_S > (1+\epsilon)d] \leq 1 - {}^{\epsilon}/_{2}$ --- high probability!
  - Thus, we repeat ${}^{8}/_{\epsilon}$ times, and take the *minimum.*
    - With ¾ probability, no trial goes below $({}^{1}/_{2} - \epsilon)\, d$, but at least one comes below $(1+\epsilon)d$. QED!

Effect of Input Query Model

- "Degree Queries"
  - Here, we assumed that we have $O(1)$ time access to degree of a node.
  - Feige's algorithm achieves $(2+\epsilon)$-approximation using $O(\sqrt{n}/\epsilon)$ queries
  - Feige also proved optimality of this algorithm: any algorithm that gives $(2-\epsilon)$-approximation must use $\Omega(n)$ queries.

- What if the query model was different?

Effect of Input Query Model

- "Neighbor Queries"
  - Query: (v, j)
  - Obtain: j^{th} neighbor of v (in some order), or "FALSE" (if v has degree less than j)
  - We can mimic degree query using $O(\log n)$ queries
    - Feige's algorithm can run using $O(\sqrt{n}\log n\ \ \epsilon^{-1})$ queries
  - Goldreich and Ron show that this model is actually very powerful
    - We can do $(1+\epsilon)$-approximation with $O\left(\sqrt{n}\,poly(\log n, \epsilon^{-1})\right)$ queries
    - They also show a $\Omega\left(\sqrt{n/\epsilon}\right)$ lower bound.

Estimating Maximal Matching

- Problem
  - Input: Graph $G = (V, E)$
  - Output: $\tilde{m}$ such that $m \leq \tilde{m} \leq m + \epsilon n$ with prob at least $^2/_3$, where m is the size of some *maximal* matching
  - Goal: $2^{O(D)}/\epsilon^2$ running time, where D is max degree
    - Sublinear time when $D = o(\log n)$

- Motivation
  - Size of maximum matching and maximum vertex cover both lie in $[m, 2m]$
  - Gives a sublinear 2-approximation algorithms for these problems

Estimating Maximal Matching

- We will estimate the size of maximal matching (MM) produced by the greedy algorithm parametrized by an ordering σ of the edges

- Greedy MM(σ):
  - Start with empty matching.
  - For $e \in E$ (in the order of σ)
    - If e does not "conflict" with already created matching, add it.

- Fix an arbitrary σ
  - We can't explicitly do this in sublinear time.
  - We'll handle this later.

Estimating Maximal Matching

- Suppose we have access to an oracle that tests whether an edge e belongs to greedy matching M.

- Algorithm:
  - $S \leftarrow {}^{8}/_{\epsilon^2}$ vertices of V sampled i.i.d.
  - X_v = 1 if there exists an edge e incident on $v \in S$ that is in M, and 0 otherwise
  - Return $\widetilde{m} = \frac{1}{2} \cdot \left(n \cdot \frac{\sum_{v \in S} X_v}{|S|}\right) + \frac{1}{2} \cdot (n \cdot \epsilon)$

Estimating Maximal Matching

- Recall: $\tilde{m} = \frac{1}{2} \cdot \left(n \cdot \frac{\sum_{v \in S} X_v}{|S|} \right) + \frac{1}{2} \cdot (n \cdot \epsilon)$
- Claim: $E[\tilde{m}] = |M| + \frac{\epsilon n}{2}$
- Proof:
  - $E\left[\frac{\sum_{v \in S} X_v}{|S|}\right]$ = prob of a random vertex being matched in M
  - $E\left[n \cdot \frac{\sum_{v \in S} X_v}{|S|}\right] = 2\ |\mathrm{M}|$ (#matched vertices = 2 $|M|$)
- To prove $|M| \le \tilde{m} \le |M| + \epsilon n$ with prob $\ge {}^{2}/_{3}$
  - Apply Hoeffding's inequality

Estimating Maximal Matching

- What's left:
  1. Design an oracle for whether e is included in M
  2. Handle the issue of not being able to fix σ beforehand
  3. Analyze running time

- Oracle: Does e belong to greedy matching M?
  - Observation: e belongs to M iff no edge e' adjacent to e with $\sigma(e') < \sigma(e)$ belongs to M.
  - Recursive call on all adjacent edges with lower priority. If all return NO, return YES, else return NO.

Estimating Maximal Matching

- What's left:
  1. Design an oracle for whether e is included in M
  2. Handle the issue of not being able to fix σ beforehand
  3. Analyze running time

- Generating permutation σ
  - We will store a random number $r_e \sim U[0,1]$ for each e.
  - We will store them in a binary search tree.
  - Start with an empty tree.
  - When we need to check the priority of e, see if it's already generated. If not, generate it.

Estimating Maximal Matching

- Running time : Oracle
  - Consider the adjacency tree for edge e.
    - Root = e
    - For every node, its children are all its adjacent edges.
  - Consider a node t at depth k
    - For the oracle to be called on t, the $k + 1$ priorities from root to t must be monotonically decreasing
    - This happens with probability $1/(k + 1)!$
  - #nodes at depth k = $(2D)^k$
    - Max degree $D \Rightarrow$ fanout is at most $2D$
  - Expected recursive calls $\leq \sum_{k=0}^{\infty} \frac{(2D)^k}{(k+1)!} \leq \frac{e^{2D}}{2D}$

Estimating Maximal Matching

- Running time : Algorithm
  - For ${}^{8}/_{\epsilon^2}$ nodes, call the oracle on all their incident edges (at most D per node)
  - Total queries to the graph = $\left({}^{8}/_{\epsilon^2}\right) \cdot D \cdot \frac{e^{2D}}{2D} = \frac{2^{O(D)}}{\epsilon^2}$
  - QED!

Estimating Maximal Matching

- Note
  - Let m^* be the size of a maximum matching
  - This only ensures $\frac{m^*}{2} \le \widetilde{m} \le 2m^* + \epsilon n$ (w.p. 2/3)
  - Suppose we want to achieve $m^* \le \widetilde{m} \le (1+\delta)m^* + \epsilon n$ (w.p 2/3)
  - Let $k = 1/\delta$
    - Nguyen and Onak show $\frac{2^{O\left(D^k\right)}}{\epsilon^{2^{k+1}}}$ query complexity
    - Yoshida, Yamamoto, and Ito improve it to $D^{O\left(k^2\right)} k^{O(k)} \epsilon^{-2}$
      - For a constant δ (thus a constant k), this is polynomial in D

Estimating Maximal Matching

- Note
  - In all the previous algorithms…
    - We ensured sublinear running time.
    - Randomization was only used to ensure that the output is approximately accurate with high probability.
  - In this algorithm…
    - We make sublinear calls to the oracle *only in expectation.* In some realizations, we might make $\Omega(n)$ oracle calls.
    - We can avoid this by "cutting off" each call to the oracle after more than $c2^{O(D)}$ recursive calls are made, for a large constant c.
    - Using Markov's inequality, this has a low chance of happening.

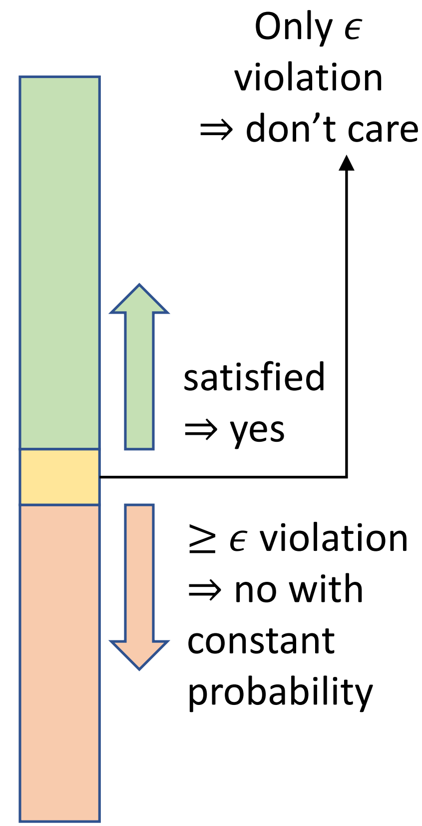
Property Testing

- The *inexact* algorithms we saw until now were about estimating numerical values.
  - I say inexact because we saw two exact algorithms for yes/no problems: binary search (deterministic) and searching in sorted list (randomized).

- We will now see inexact algorithms for yes/no problems.
  - One such area is "property testing".
  - It's one of the most prevalent applications of sublinear time algorithms, and a research area of its own.

Property Testing

- **Problem:**
  - Given input I, test if it satisfies property P.
- **Inexact goal:**
  - If I satisfies P, must return "yes".
  - If I is at least "ϵ-far" from satisfying P, must return "no" with probability at least $^2/_3$.
  - If I violates P, but is "ϵ-close" to satisfying P, free to return anything (we don't care!).
- Notes
  - For 2-sided error, we also require "yes" w.p. at least $^2/_3$.
  - What's "ϵ-far"? We'll see.



Testing Linearity of Function

- Consider a Boolean function $f: \{0,1\}^n \to \{0,1\}$
- We want to test if f is *linear*:
  - $\exists a_1, \dots, a_n \in \{0,1\}$ s.t. $f(x_1, \dots, x_n) = a_1 x_1 + \cdots + a_n x_n$?
  - All computations are in $\mathbb{F}_2$ (modulo 2).
  - Equivalently: $f(x+y) = f(x) + f(y),\ \forall x, y \in \{0,1\}^n$?

- We say that f is ϵ-close to being linear if $\exists g$ such that $|\{x : f(x) \neq g(x)\}| \le \epsilon 2^n$.
  - Only need to change ϵ fraction of values to make it linear.

Testing Linearity of Function

- Input: Oracle for accessing f
- Goal: 1-sided algorithm for testing linearity of f that makes $O(1/\epsilon)$ queries.
  - Note: This is independent of n. This is actually achievable for testing many properties.
- Motivation
  - Subroutine for many other property testing algorithms
  - Applications in cryptography, coding theory, program checking, PCPs (inapproximability), and Fourier analysis

Testing Linearity of Function

- Algorithm:
  - Sample $^{2}/_{\epsilon}$ random pairs (x, y)
  - If $f(x+y) \neq f(x) + f(y)$ for any pair, output "no".
  - Else, output "yes".

- Note
  - Algorithm always outputs "yes" if f is linear.
  - We want to prove that if f is ϵ-far from being linear, then it outputs "no", i.e., finds a "violating pair" with probability at least $^{2}/_{3}$.

Testing Linearity of Function

- [Bellare, Coppersmith, Hastad, Kiwi, Sudan '95]
  If f is ϵ-far from linear, then the test fails on a random (x, y) pair with probability at least ϵ.
  - Deep result that uses results from Fourier analysis.

- Assuming this result…
  - Probability that algorithm fails on 1 sample $\leq 1 - \epsilon$
  - Probability that algorithm fails on $2/\epsilon$ samples $\leq$ $(1-\epsilon)^{\frac{2}{\epsilon}} \leq \left(\frac{1}{e}\right)^2 < \frac{1}{3}$